Learning Pauli Commuting Local Hamiltonians

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Abstract

Learning an unknown Hamiltonian from local measurements is an increasingly important task in the NISQ era. Recent work by [BAL19] proposed an approach to learn *non-commuting* local Hamiltonians, though their method fails for *commuting* Hamiltonians.

We provide a method to learn *Pauli commuting local Hamiltonians*, which is a subclass of general CLHs. Given $\exp(n)$ copies of the Gibbs state ρ of a Pauli commuting local Hamiltonian H on n qubits, one can learn such a Hamiltonian by

1. Applying a *linear-depth* Clifford circuit on given copies;
2. Performing classical post-processing.

Our result sheds light on learning general commuting local Hamiltonians using local measurements.

Problem Statement: A Quantum Perspective

Consider an "inverse problem" of finding ground states by given a Hamiltonian (i.e., the *local Hamiltonian problem*), namely

m copies of the Gibbs state ρ → Leaning a local Hamiltonian by an algorithm $\mathcal{A}$ → Coefficients of a local Hamiltonian

Q1: What's the *sample complexity* m?

Q2: What's the *time complexity* of a learning algorithm $\mathcal{A}$?

Problem Statement: A Classical Perspective

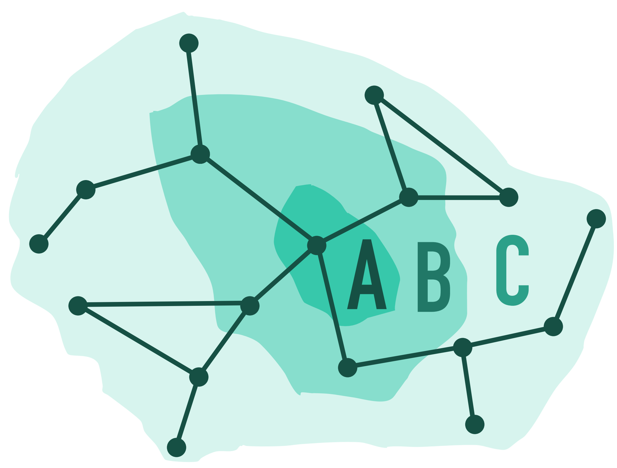
The classical analog is learning a Markov random field (e.g. [KM17]):

Gibbs distribution defined on $z = (z_1, \cdots, z_n) \in \{\pm 1\}^n$,

$$\Pr[Z = z] \propto \exp(-H_c) := \exp\left(\sum_{i \neq j \in [n]} A_{ij} z_i z_j + \sum_{i \in [n]} \theta_i z_i\right).$$

Configuration graph $G = ([n], E)$ where $(i,j) \in E$ if $A_{i,j} \neq 0$.

Let $\mathbf{X_i} := \{z_j | j \neq i, (i,j) \in E\}$ and $Y_i := (1 - z_i)/2$ be random variables. Note $\mathbf{X_i}$ is only dependent on *the neighbors of* i due to the *Markovianity* $\Pr[A|B] = \Pr[A, C|B]$.



Task. Given m random samples $(\mathbf{X_i}, Y_i)$ satisfying $\mathbb{E}[Y_i | \mathbf{X_i} = \mathbf{x}] = \sigma(\mathbf{A_j} \cdot \mathbf{x} + \theta_\mathbf{i})$ where $\sigma(x) = 1/(1 + e^{-x})$, recover $\mathbf{A_j}$ and $\theta_\mathbf{i}$.

Main Algorithm: Learning a Pauli CLH

m copies of the Gibbs state of Pauli CLH

Mapping Gibbs states of a Pauli CLH into Gibbs states of a classical Hamiltonian [AG04,JST+20]

Learning candidates of local terms in Hamiltonian using O(log n) samples [BAL19,CW19]

Draw samples from the classical Gibbs distribution

local measurements

Classical post-processing with exp(n) samples: Learning a classical Hamiltonian

[KM17]

Mappling a classical Hamiltonian back into a Pauli CLH

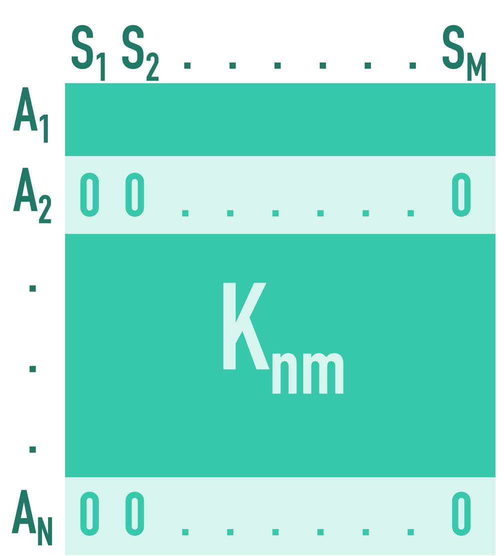
[AG04,JST+20]

Coefficients (associated with local terms) of a Pauli CLH

Proof Technique: Learning candidates of local terms

Consider $H_L = \sum_{m=1}^{M} c_m S_m$ defined on a local patch L with a Gibbs state ρ, [BAL19] implies that $\forall$ local observables A_n inside L, $\sum_{m=1}^{M} c_m \mathrm{Tr}(\mathbf{i}\rho[S_m, A_n]) = 0$ and $|\{A_n \text{ on } L\}| \leq \mathrm{poly}(n)$. Let matrix $K_{nm} := \mathrm{Tr}(\mathbf{i}\rho[S_m, A_n])$.

Claim. If A_n is a local term in H_L, then $\forall S_m$, $[S_m, A_n] = 0$, i.e., the n-th row of K is all-zero.



Defintion: Pauli *Commuting* Local Hamiltonians

A k-CLH $H = \sum_{i=1}^{m} H_i$ on n-qubit is a Pauli CLH if it satisfies

- $\forall i \in [m]$, $H_i = a_i \sigma_1^{(i)} \otimes \cdots \otimes \sigma_n^{(i)}$, where $\sigma_j^{(i)} := \mathbf{i}^{x_j^{(i)} z_j^{(i)}} X^{x_j^{(i)}} Z^{z_j^{(i)}}$.
- $\forall i, j \in [m]$, $[H_i, H_j] = 0$.

A Pauli CLH can be described by *a Stabilizer tableau* [AG04], namely each local term can be represented by a $(2n+1)$-tuple:

	a_i	$x_1^{(i)}$	$\cdots$	$x_n^{(i)}$	$z_1^{(i)}$	$\cdots$	$z_n^{(i)}$	
H_1	a_1	$x_1^{(1)}$	$\cdots$	$x_n^{(1)}$	$z_1^{(1)}$	$\cdots$	$z_n^{(1)}$	$\mathbf{Z}^{(1)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
H_m	a_m	$\mathbf{X}^{(m)}$ $x_1^{(m)}$	$\cdots$	$x_n^{(m)}$	$z_1^{(m)}$	$\cdots$	$z_n^{(m)}$	
	(p)		(A)	$\mathbf{X}_n$	$\mathbf{Z}_1$	(B)		

Commutation. $\forall i, j$, $[H_i, H_j] = 0 \Leftrightarrow \mathbf{X}^{(i)} \cdot \mathbf{Z}^{(j)} \oplus \mathbf{X}^{(j)} \cdot \mathbf{Z}^{(i)} = 0$.

Linear *column* operations on the tableau. It is equivalent to apply Clifford gates [AG04], such as Hadamard, S, CNOT:

- Had_k: swap $\mathbf{X}_k$ with $\mathbf{Z}_k$ and $\mathbf{p}' := \mathbf{p} \oplus (\mathbf{X}_i \odot \mathbf{Z}_i)$.
- S_k: $\mathbf{Z}'_k := \mathbf{Z}_k \oplus \mathbf{X}_k$ and $\mathbf{p}' := \mathbf{p} \oplus (\mathbf{X}_i \odot \mathbf{Z}_i)$.
- $\mathrm{CNOT}_{i,j}$: $\mathbf{X}'_j := \mathbf{X}_i \oplus \mathbf{X}_j$, $\mathbf{Z}'_i := \mathbf{Z}_i \oplus \mathbf{Z}_j$ and $\mathbf{p}' := \mathbf{p} \oplus (\mathbf{X}_i \odot \mathbf{Z}_j \odot (\mathbf{X}_j \oplus \mathbf{Z}_i \oplus \mathbf{1}))$.

Proof Technique: Mapping Pauli CLHs into Classical Hamiltonians

Applying a $O(n/\log n)$-depth Clifford circuit [AG04, JST+20]:

1) Gaussian Elimin.: $\left(\begin{array}{c|c|c} p & A & B \end{array}\right) \xrightarrow{\text{CNOT}} \left(\begin{array}{c|c|c} p^{(1)} & I & B_1 \\ & A_2 & B_2 \end{array}\right)$;

2) Making Full-rank.: $\xrightarrow{\text{S}} \left(\begin{array}{c|c|c} p^{(2)} & I & B_1^{(1)} \\ & A_2 & B_2^{(1)} \end{array}\right) = \left(\begin{array}{c|c|c} p^{(2)} & I & NN^T \\ & A_2 & B_2^{(1)} \end{array}\right)$;

3) Cholesky Decom: $\xrightarrow{\text{CNOT}} \left(\begin{array}{c|c|c} p^{(3)} & N & N \\ & A_2^{(1)} & B_2^{(2)} \end{array}\right) \xrightarrow{\text{S}} \left(\begin{array}{c|c|c} p^{(4)} & 0 & N \\ & A_2^{(2)} & B_2^{(2)} \end{array}\right)$;

4) Gaussian Elimin.: $\xrightarrow{\text{CNOT}} \left(\begin{array}{c|c|c} p^{(5)} & 0 & I \\ & A_2^{(3)} & B_2^{(3)} \end{array}\right) = \left(\begin{array}{c|c|c} p^{(5)} & 0 & I \\ & 0 & B_2^{(3)} \end{array}\right)$.

An efficient classical post-processing condition:

$\mathrm{rank}(A|B) = m$ and all rows in $(A|B)$ are *linearly independent*.

Now we obtain a 1-local classical Hamiltonian since $B_2^{(3)} = 0$. It can be learned *efficiently* in both time complexity and sample complexity.

Open Problem: Towards an Efficient Classical Post-processing

A classical algorithm for learning classical Hamiltonians. [KM17] provides an algorithm for learning a k-local classical Hamiltonian with run-time $n^{\Theta(k)}$ and sample complexity $n^{O(k)}$.

Main issues. The resulting classical Hamiltonian H' is *non-necessarily local* since $B_2^{(3)} \neq 0$ in general, so applying [KM17] for H' directly requires $\exp(n)$ run-time and $\exp(n)$ samples. Could we learn a classical Hamiltonian obtained from a Pauli CLH *efficiently*?

Open Problem: Learning CLHs by Matrix MWU Methods

[KM17] is based on *multiplicative weight updates (MWU)* and *Markov property* of Gibbs distributions. Notice Markov property holds for commuting local Hamiltonians due to the *Koashi-Imoto decomposition* [KI02]. Could we learn CLHs using *Matrix* MWU methods?

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