Computational hardness of estimating quantum entropies via binary entropy bounds

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1. Quantum state testing with respect to different entropy measures
2. Main results: Computational hardness of estimating entropies of rank-2 states
3. Proof techniques
4. Open problems

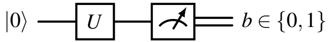
What is quantum state testing

Basic ingredients in quantum computation:

- **Quantum states.** An n-qubit quantum state $\rho \in \mathbb{C}^{N \times N}$, where $N = 2^n$, is an N-dimensional positive semi-definite (PSD) matrix such that $\mathrm{Tr}(\rho) = 1$.
- **Pure states.** An n-qubit state is *pure* if $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle \in \mathbb{C}^N$ and $\langle\psi|\psi\rangle = 1$.
  In the single-qubit case, $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ such that $|\alpha|^2 + |\beta|^2 = 1$, $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.
- **Purification**. For any n-qubit quantum state ρ on $\mathcal{H}_\mathsf{A}$, there exists a $2n$-qubit pure state $|\psi\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $\mathrm{Tr}_\mathsf{B}(|\psi\rangle\langle\psi|) = \rho$.
  For instance, let $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then $\mathrm{Tr}_2(|\phi\rangle\langle\phi|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = I/2$.
- **Quantum gate**. Elementary quantum gates G_i (from some universal gateset) are unitary matrices act on one or two qubits, e.g., $G_i \in \{\mathsf{CNOT}, \mathsf{Had}, \mathsf{T}\}$:

$$|0\rangle^{\otimes n} \xrightarrow{G_1} G_1\,|0\rangle^{\otimes n} \xrightarrow{G_2} G_2 G_1\,|0\rangle^{\otimes n} \rightarrow \cdots$$

- **Measurement**. Projective measurement in computational basis $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$:



Task: Quantum state testing via entropy approximation

Given a state-preparation circuit Q ("quantum devices") that prepares (the purification of) n-qubit quantum states $\rho \in \mathbb{C}^{N \times N}$. Decide whether $\mathrm{Ent}(\rho) \geq \tau_0(n)$ or $\mathrm{Ent}(\rho) \leq \tau_1(n)$.

What is quantum state testing (Cont.)

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- Quantum devices Q can be given either as a query oracle (*black-box model*) or a sequence of $\mathrm{poly}(n)$ elementary quantum gates (*white-box model*).
- The most canonical choices of entropy measures are:
  - **von Neumann entropy** $\mathrm{S}(\rho) := -\mathrm{Tr}(\rho \ln \rho)$.
  - **Shannon entropy** $\mathrm{H}(D) := \sum_x -D(x) \ln D(x)$.
- Entropy *difference* problems, with respect to the quantity $\mathrm{Ent}(\rho_0) - \mathrm{Ent}(\rho_1)$, can be defined similarly and ask whether

$$\mathrm{Ent}(\rho_0) - \mathrm{Ent}(\rho_1) \geq g(n) \quad \text{or} \quad \mathrm{Ent}(\rho_0) - \mathrm{Ent}(\rho_1) \leq -g(n).$$

Typical goal. Minimize the "complexity" of ρ (or its corresponding Q):

Type of query access	Complexity measure
Black-box model	Query complexity (the number of queries to Q)
White-box model	Complexity class

Generalizations of the von Neumann entropy

<u>**Generalizations**</u>. There are two families of generalizations of the von Neumann entropy $\mathrm{S}(\rho)$, namely, the α-Rényi entropy $\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho)$ and the q-Tsallis entropy $\mathrm{S}^{\mathtt{T}}_{q}(\rho)$:

$$\mathrm{S}^{\mathtt{R}}_{\alpha}(\rho) := \frac{\ln \mathrm{Tr}(\rho^{\alpha})}{1-\alpha} \quad \text{and} \quad \mathrm{S}^{\mathtt{T}}_{q}(\rho) := \frac{1-\mathrm{Tr}(\rho^{q})}{q-1}.$$

As the order approaches 1, these two generalizations converge to $\mathrm{S}(\rho)$.

<u>**von Neumann entropy (order 1)**</u>. The entropy approximation problem in this case is *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}(\rho)$ to within additive error ε is $\widetilde{O}(r/\varepsilon^2)$ [Wang–Guan–Liu–Zhang–Ying'22] and $\Omega\big(\frac{\sqrt{r}}{\sqrt{\varepsilon}} + \frac{\ln r}{\varepsilon}\big)$ [Chen–Wang–Zhang'25].
- The promise problem QUANTUM ENTROPY APPROXIMATION (QEA), with respect to $\mathrm{S}(\rho)$, is NIQSZK-complete [Kobayashi'02, Chailloux–Ciocan–Kerenidis–Vadhan'07].
  - It is widely believed that BQP $\subsetneq$ NIQSZK.
- The *poly(n)-rank* variant LOWRANKQEA is BQP-complete: containment in [Wang–Guan–Liu–Zhang–Ying'22] and hardness in [**L.**–Wang'24].
  - <u>Containment</u>: a polynomial-time ("efficient") quantum algorithm that solves the problem.
  - <u>Hardness</u>: the problem requires *at least* the full power of a quantum computer to solve efficiently; namely, if you can solve this problem, you can solve all problems in BQP.

Generalizations of the von Neumann entropy (Cont.)

Prior quantum query complexity upper bounds are summarized as follows:

Order (α or q)	Quantum α-Rényi entropy	Quantum q-Tsallis entropy
$(0,1)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]
1	$\mathrm{poly}(r, 1/\varepsilon)$ [WGLZY22]	
$(1,\infty)$	$\mathrm{poly}(r, 1/\varepsilon)$ [WZL24]	$\mathrm{poly}(1/\varepsilon)$ [**L.**–Wang'24]

We also summarize the prior work in terms of complexity classes:

- ⋆ For all $\alpha > 0$, LowRankRényiQEA$_\alpha$ is in BQP [Wang–Zhang–Li'22].
- ⋆ For all $q \in (0,1)$, LowRankTsallisQEA$_q$ is in BQP [WGLZY22].
- ⋆ For the order-1 case (von Neumann entropy), LowRankQEA is BQP-complete; containment follows from [WGLZY22], and hardness from [**L.**–Wang'24].
- ⋆ For all $q \in (1,2]$, TsallisQEA$_q$ is BQP-complete [**L.**–Wang'24].
- ⋆ For all $q > 2$, TsallisQEA$_q$ is in BQP [**L.**–Wang'24].

These results lead to the following questions:

1. How hard is the task of estimating α-Rényi or q-Tsallis entropy of quantum states for *all* positive order α or q?
2. Could LowRankRényiQEA$_\alpha$ $(\alpha > 0)$ and LowRankTsallisQEA$_q$ $(q > 0)$ both be BQP-hard, thus capturing the full power of quantum computation?

Main results

Let $\textsc{Rank2RényiQEA}_\alpha$ and $\textsc{Rank2TsallisQEA}_q$ denote the restricted versions of $\textsc{RényiQEA}_\alpha$ and $\textsc{TsallisQEA}_q$, where the state ρ has rank 2.

Theorem 1 (Hardness of estimating quantum entropies with positive orders).

1. For all real-valued $\alpha > 0$ and $\alpha = \infty$, $\textsc{Rank2RényiQEA}_\alpha$ is BQP-hard.
2. For all real-valued $q > 0$, $\textsc{Rank2TsallisQEA}_q$ is BQP-hard.

Combining Theorem 1 with prior quantum query complexity upper bounds implies:

Corollary 2. For all real-valued $\alpha > 0$, $\textsc{LowRankRényiQEA}_\alpha$ is BQP-complete.

Corollary 3. The following holds:

1. For all real-valued $q \in (0, 1]$, $\textsc{LowRankTsallisQEA}_q$ is BQP-complete.
2. For all real-valued $q > 1$, $\textsc{TsallisQEA}_q$ is BQP-complete.

Theorem 4. $\textsc{Rank2RényiQEA}_0$ and $\textsc{Rank2TsallisQEA}_0$ are NQP-complete.

- NP $\subseteq$ NQP by comparing definitions, while it is widely believed that NP $\not\subseteq$ BQP.

The BQP-hardness in Theorem 1 holds for the *smallest non-trivial* rank, as rank-1 states are pure states whose entropies are 0 for all orders.

Prior approaches via quantum entropy difference

The key quantity behind the prior approaches in [Ben-Aroya–Ta-Shma'07, **L.**'23] is the *quantum Jensen–Shannon divergence* $\mathrm{QJS}(\rho_0,\rho_1)$, introduced in [Majtey–Lamberti–Prato'05]:

$$\mathrm{QJS}(\rho_0,\rho_1) := \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2} = \frac{1}{2}\cdot\left(\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\otimes\frac{\rho_0+\rho_1}{2}\right) - \mathrm{S}(\rho_0\otimes\rho_1)\right).$$

The BQP-hardness of LowRankQED (and LowRankQEA) follows from the facts:

1. The *pure-state variant* of the Quantum State Distinguishability Problem (PureQSD$[2/3,1/3]$), deciding whether $\mathrm{T}(|\psi_0\rangle\langle\psi_0|,|\psi_1\rangle\langle\psi_1|)$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21, Wang–Zhang'23].
2. The inequalities connect QJS to T [Fuchs–van de Graaf'99, Briët–Harremoës'09]:

$$\mathrm{H}\left(\frac{1}{2}\right) - \mathrm{H}\left(\frac{1-\mathrm{T}(\rho_0,\rho_1)}{2}\right) \leq \mathrm{QJS}(\rho_0,\rho_1) \leq \mathrm{H}\left(\frac{1}{2}\right)\cdot\mathrm{T}(\rho_0,\rho_1).$$

This approach can be generalized to TsallisQED$_q$ **($1\leq q\leq 2$).**

The key quantity is the *quantum Jensen–Tsallis divergence* $\mathrm{QJT}_q(\rho_0,\rho_1)$ for $1\leq q\leq 2$, introduced in [Briët–Harremoës'09], whose square root is a metric [Virosztek'19, Sra'19]:

$$\mathrm{QJT}_q(\rho_0,\rho_1) := \mathrm{S}_q^{\mathtt{T}}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}_q^{\mathtt{T}}(\rho_0)+\mathrm{S}_q^{\mathtt{T}}(\rho_1)}{2}.$$

Using the *joint convexity* of QJT_q [Chen–Tropp'13, Virosztek'19], [**L.**–Wang'24] establishes:

$$\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) - \mathrm{H}_q^{\mathtt{T}}\left(\frac{1-\mathrm{T}(\rho_0,\rho_1)}{2}\right) \leq \mathrm{QJT}_q(\rho_0,\rho_1) \leq \mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right)\cdot\mathrm{T}(\rho_0,\rho_1)^q.$$

Establishing the hardness via binary entropy bounds

We start with a new approach that establishes the BQP-hardness of RANK2QEA. This approach is based on two key observations:

1. The 2-Tsallis entropy of a rank-2 state $\frac{1}{2}(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|)$ can be expressed as the 2-Tsallis binary entropy, and this coincidence naturally extends to all $q > 0$:
$$\mathrm{S}_2^{\mathtt{T}}\left(\frac{|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|}{2}\right) = \frac{1 - |\langle\psi_0|\psi_1\rangle|^2}{2} = \mathrm{H}_2^{\mathtt{T}}\left(\frac{1 - \langle\psi_0|\psi_1\rangle}{2}\right).$$
2. The following bounds on the Shannon binary entropy in [Lin'91, Topsøe'01]:
$$2\mathrm{H}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}(x) \leq \sqrt{2}\mathrm{H}\left(\frac{1}{2}\right) \cdot \sqrt{\mathrm{H}_2^{\mathtt{T}}(x)}.$$

Since PURE-STATE INFIDELITY ESTIMATION (PUREINFIDELITY$\left[\frac{2}{3}, \frac{1}{3}\right]$), deciding whether $1 - |\langle\psi_0|\psi_1\rangle|^2$ is at least $2/3$ or at most $1/3$, is BQP-hard [Rethinasamy–Agarwal–Sharma–Wilde'21], it follows that both RANK2TSALLISQEA$_2$ and RANK2RÉNYIQEA$_2$ are BQP-hard.

The inequalities relating the order-2 binary entropy to Shannon binary entropy (Step 2) can be extended to *all* positive orders! This is the main technical contribution of our work and explains why rank-2 quantum states suffice to capture the BQP-hardness.

Establishing the hardness via binary entropy bounds (Cont.)

The BQP-hardness of RANK2RÉNYIQEA$_\alpha$ can then be established via the following inequalities that relate $\mathrm{H}_2^{\mathrm{R}}(x)$ to $\mathrm{H}_\alpha^{\mathrm{R}}(x)$:

Range of α	Hardness	Reduction from	New inequalities
$0<\alpha<1$	BQP-hard Theorem 1(1)	RANK2RÉNYIQEA$_2$	$\mathrm{H}_2^{\mathrm{R}}(x) \leq \mathrm{H}_\alpha^{\mathrm{R}}(x)$ $\mathrm{H}_\alpha^{\mathrm{R}}(x) \leq \ln(2)^{1-\frac{\alpha}{2}} \cdot \mathrm{H}_2^{\mathrm{R}}(x)^{\frac{\alpha}{2}}$ [Beck–Schögl'93, Sec 5.3]
$1 \leq \alpha < 2$	BQP-hard Theorem 1(1)	RANK2RÉNYIQEA$_2$	
$\alpha = 2$	BQP-hard Theorem 1(1)	PUREINFIDELITY [RASW'21]	None
$\alpha \in (2,\infty]$	BQP-hard Theorem 1(1)	RANK2RÉNYIQEA$_2$	$\frac{\alpha}{2(\alpha-1)} \cdot \mathrm{H}_2^{\mathrm{R}}(x) \leq \mathrm{H}_\alpha^{\mathrm{R}}(x) \leq \mathrm{H}_2^{\mathrm{R}}(x)$ [Beck–Schögl'93, Sec 5.3]

Establishing the hardness via binary entropy bounds (Cont.^{2})

The BQP-hardness of RANK2TSALLISQEA$_q$ can then be established via the following inequalities that relate $\mathrm{H}_2^{\mathtt{T}}(x)$ to $\mathrm{H}_q^{\mathtt{T}}(x)$:

Range of q	Hardness	Reduction from	New inequalities
$0 < q < 1$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x)$ $\mathrm{H}_q^{\mathtt{T}}(x) \leq 2^{\frac{q}{2}}\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \left(\mathrm{H}_2^{\mathtt{T}}(x)\right)^{\frac{q}{2}}$ [L.–Wang'24]
$1 \leq q < 2$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	
$q = 2$	BQP-hard Theorem 1(2)	PUREINFIDELITY [RASW'21]	None
$2 < q \leq 3$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$\frac{q}{2(q-1)} \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x) \leq 2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x)$
$q \in (3, \infty)$	BQP-hard Theorem 1(2)	RANK2TSALLISQEA$_2$	$2\mathrm{H}_q^{\mathtt{T}}\left(\frac{1}{2}\right) \cdot \mathrm{H}_2^{\mathtt{T}}(x) \leq \mathrm{H}_q^{\mathtt{T}}(x)$ $\mathrm{H}_q^{\mathtt{T}}(x) \leq \frac{q}{2(q-1)} \cdot \mathrm{H}_2^{\mathtt{T}}(x)$ [L.–Wang'24]

The additional row links to the *normalized* q-Tsallis entropy $\widetilde{H}_q^{\mathtt{T}}(x) := \mathrm{H}_q^{\mathtt{T}}(x)/\mathrm{H}_q^{\mathtt{T}}(1/2)$ (cf. [Daróczy'70]), whose monotonicity *changes* at some point $q^*(x) \in [2,3]$.

Conclusions and open problems

Take-home messages on our work

For ***all*** **positive** orders, estimating α-Rényi or q-Tsallis entropies of **rank-**2 quantum states, *the smallest non-trivial rank*, is BQP-hard.

Discussion and open problems

Two limitations of our new approach are as follows:

- Our approach works only when quantum entropy values and the promise gap $\tau_0 - \tau_1$ are both *constant*. Otherwise, reductions based on inequalities, such as $\mathrm{S}^{\mathrm{R}}_{\infty}(\rho) \leq \mathrm{S}^{\mathrm{R}}_{\alpha}(\rho) \leq \frac{\alpha}{\alpha-1}\mathrm{S}^{\mathrm{R}}_{\infty}(\rho)$ for all $\alpha > 1$, break down for sufficiently large n.
- A complexity-theoretic barrier is that reductions between different orders do not hold in general. Noting that RÉNYIQEA$_{\infty}$ is coSBP-complete [Watson'12] and EA is NISZK-complete [Goldreich–Vadhan'99], such reductions would yield

  $$\text{coNP} \subseteq \text{coSBP} \subseteq \text{NISZK} \subseteq \text{SZK} \subseteq \text{AM} \cap \text{coAM}.$$

  These inclusions would collapse PH to its second level [Boppana–Håstad–Zachos'87].

Question: Is it possible to establish a complexity-theoretic classification theorem for estimating the quantum (α-Rényi or q-Tsallis) entropies?

Thanks!

Generalizations of the von Neumann entropy: Prior work

α-Rényi entropy ($\alpha > 0$). The corresponding entropy approximation problems are also *hard*, with complexity depending (polynomially) on the rank r of the state:

- The query complexity for estimating $\mathrm{S}^{\mathrm{R}}_{\alpha}(\rho)$ to within additive error ε is $\widetilde{O}(r^{\frac{1}{\alpha}}/\varepsilon^{1+\frac{1}{\alpha}})$ for $0 < \alpha < 1$ and $\widetilde{O}(r/\varepsilon^{1+\frac{1}{\alpha}})$ for $\alpha > 1$ [Wang–Zhang–Li'22], while the lower bounds depend polynomially on r and $1/\varepsilon$ (e.g., [Wang–Guan–Liu–Zhang–Ying'22]).
- For $\alpha \in (0,1) \cup (1,\infty)$, the *low-rank* variant of the QUANTUM α-RÉNYI ENTROPY APPROXIMATION (LOWRANKRÉNYIQEA$_\alpha$) is in BQP [Wang–Zhang–Li'22].

q-Tsallis entropy ($0 < q < 1$). The entropy approximation problems are *hard*:

- For all $q \in (0,1)$, the query complexity for estimating $\mathrm{S}^{\mathrm{T}}_{q}(\rho)$ to within additive error ε is $\mathrm{poly}(r, 1/\varepsilon)$ [Wang–Guan–Liu–Zhang–Ying'22], specifically $\widetilde{O}(r^{\frac{3-q^2}{2q}}/\varepsilon^{\frac{3+q}{2q}})$.
- For all $q \in (0,1)$, the *low-rank* variant of the QUANTUM q-TSALLIS ENTROPY APPROXIMATION (LOWRANKTSALLISQEA$_q$) is in BQP [Wang–Guan–Liu–Zhang–Ying'22].

q-Tsallis entropy ($q > 1$). The corresponding entropy approximation problems are *easy*, with *rank-independent* complexity:

- The query complexity for estimating $\mathrm{S}^{\mathrm{T}}_{q}(\rho)$ is $O(1/\varepsilon^{1+\frac{1}{q-1}})$ for $q > 1$ [**L.**–Wang'24], while it is $\Omega(1/\varepsilon^{\frac{1}{2(q-1)}})$ for $1 < q < \frac{3}{2}$ and $\Omega(1/\varepsilon)$ for $q \geq \frac{3}{2}$ [Chen–Wang–Zhang'25].
- For all $q \in (1,2]$, TSALLISQEA$_q$ and TSALLISQED$_q$ are BQP-complete, while the BQP containment also holds for $q > 2$ [**L.**–Wang'24].