Complexity Zoo and Local Hamiltonian Problem

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Apr 19, 2017

Outline

Part I: Complexity Zoo

- Decision problem and language
- Circuit model (logic gate and quantum gate)
- Complexity classes: P and BQP
- More complexity classes: NP, QMA and #P
- Reduction

Part II: Local Hamiltonian Problem

- Local Hamiltonian and LHP
- How hard is the local Hamiltonian?
- An approach to show a class of LHP in NP(P)
- $1D$ gapped LHP is in P
- Is $2D$ gapped LHP in NP?

Preliminary

Decision problem $f : \{0,1\}^* \to \{0,1\}$

Counting problem $f : \{0,1\}^* \to \mathbb{N}$

Language $L = \{s \in \{0,1\}^* = \cup_{k=1}^{\infty} \{0,1\}^k : f(s) = 1\}$

Problem size(input size) # input bits

Time # gates in circuit

decide the language = solving decision problem

e.g. Factoring

Input: $n, k \in \mathbb{N}$

Output:

- Yes if n has factor $< k$;
- No, otherwise.

Notice that the problem size(input size) is $\log(n)$.

Circuit Model: logic & quantum gates

Logic gate

Boolean function on 1 or 2 bits.

$$G_1 : \{0,1\} \to \{0,1\}$$
$$G_2 : \{0,1\}^{\times 2} \to \{0,1\}$$

Quautum gate

Unitary operator on 1 or 2 qubits:

$$U_1 : \mathbb{C}^2 \to \mathbb{C}^2$$
$$U_2 : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$$
(where $U^\dagger U = \mathbb{I}$)

e.g.

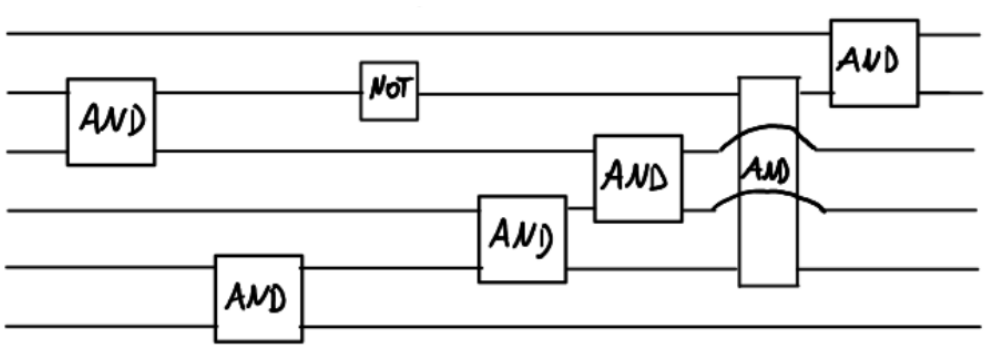
$$AND(x,y) = \begin{cases} 1, & x = y = 1 \\ 0, & \text{otherwise} \end{cases}$$

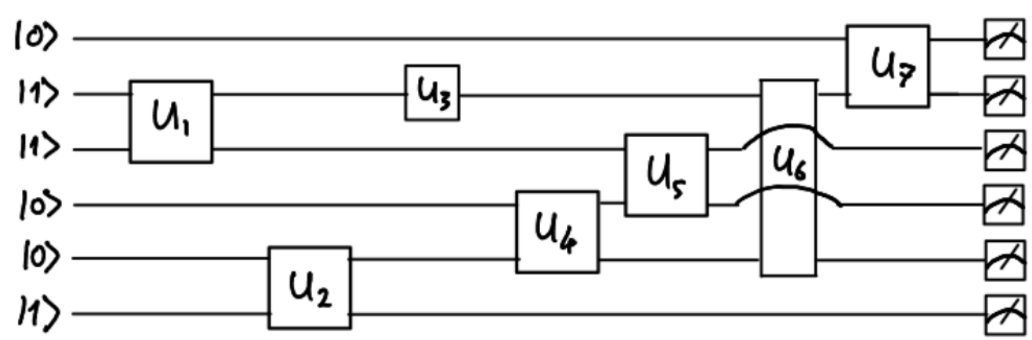
$$NOT(x) = \begin{cases} 0, & x = 1 \\ 1, & x = 0 \end{cases}$$

e.g. **Hadamard** $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + |1\rangle\langle 1| \otimes (|1\rangle\langle 0| + |0\rangle\langle 1|)$$





Circuit Model: Quantum Circuit

Quantum circuit

finite sequence of quantum gates.

Input $|\psi\rangle$ is "computational basis" state, i.e. $|\psi\rangle = \otimes_i |x_i\rangle$, $|x_i\rangle \in \{|0\rangle, |1\rangle\}$.

Output outcome of measuring qubits in computational basis.

i.e. measure $\{\Pi^{(0)}, \Pi^{(1)}$ on each qubit, where $\mathrm{Tr}(\Pi^{(i)}\rho)$.

e.g. Universal set

Clifford gate:

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$

Phase $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = |0\rangle\langle 0| + i|1\rangle\langle 1|$

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + |1\rangle\langle 1| \otimes (|1\rangle\langle 0| + |0\rangle\langle 1|)$$

T-gate $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

Complexity classes: P and BQP

Uniform circuit family Sequence of circuits C_n on n-bit is uniform if $\exists$ Turing Machine which on input n, outputs description of C_n in poly-time.

P **(Polynomial time)** Class of decision problems solvable with uniform family of poly-sized circuit.

e.g.
- Frustration-free 2-LHP $\in$ P [Bravyi '06]
- Determinant $\in$ P

BQP **(Bounded-error quantum poly. time, so-called "quantum P")** $\exists$ poly-sized uniform quantum circuit U s.t.

$$\Pr(U \text{ outputs "1"}) = \begin{cases} \geq \frac{2}{3}, & \text{Yes instance} \\ \leq \frac{1}{3}, & \text{No instance} \end{cases}$$

where $\Pr(U \text{ outputs "1"}) = \mathsf{Tr}[\Pi_1^{(1)} \otimes \mathbb{I}_{2\cdots n} U|x\rangle\langle x|U^\dagger] = \langle x|U^\dagger \Pi_1^{(1)} \otimes \mathbb{I}_{2\cdots n} U|x\rangle$ and the output is given by the first qubit.

e.g.
- Factoring $\in$ BQP [Shor '94]
- All equivalent quantum computing model is BQP-complete, such as topological(Jones polynomial), adabatic, $\cdots$

Examples: k-fold Forrelation

k-fold Forrelation

Given boolean functions $f_1, \cdots, f_k : \{0,1\}^n \to \{0,1\}^n$, its k-fold forrelation is the following quality:

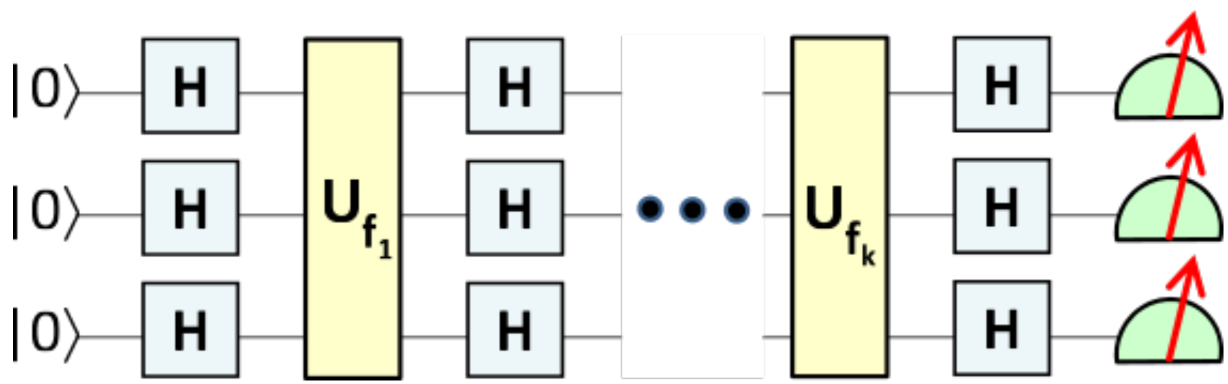
$$\Phi_{f_1,\cdots,f_k} = \frac{1}{2^{(k+1)n/2}} \sum_{x_1,\cdots,x_k \in \{0,1\}} (-1)^{f_1(x_1)}(-1)^{x_1\cdot x_2}(-1)^{f_2(x_2)}\cdots(-1)^{x_{k-1}\cdot x_k}(-1)^{f_k(x_k)}$$

Input k Boolean circuits $C_1, \cdots, C_k$, which compute the Boolean functions $f_j : \{0,1\}^n \to \{0,1\}^n$.

Promise either $\Phi_{f_1,\cdots,f_k} \leq \frac{1}{100}$ or $\Phi_{f_1,\cdots,f_k} \geq \frac{3}{5}$.

Output decide whether $\Phi_{f_1,\cdots,f_k} > \frac{1}{2}$.

- $x_1 \cdot x_2$ is dot product.
- $H^{\otimes n}|0\rangle^{\otimes n} = 2^{-n/2} \sum_{x_1 \in \{0,1\}^n} |x_1\rangle$
- $H^{\otimes n} f_1(x_1) \sum_{x_1} |x_1\rangle = 2^{-n} \sum_{x_1,x_2} (-1)^{x_1\cdot x_2} f_1(x_1)|x_2\rangle$



k-fold Forrelation$(k = poly(n))$ is BQP-complete [Aaronaon&Ambainis '14]

Complexity classes: NP and QMA

NP **(non-deterministic poly. time)** Class of decision problems for which ∃ polynomial time verifier V s.t. if answer for input x is

- Yes: ∃ polynomial-sized "witness" w s.t. $V(x, w) = 1$.
- No: ∀ witness w, $V(x, w) = 0$.

Merlin ☥ — *convince* → Arthur ♔

Merlin:
- all-powerful
- untrustworthy

Arthur:
- poly-time computation

QMA **(quantum Merlin-Arthur, so-called "quantum NP")** ∃ poly-sized uniform quantum verifier circuit U s.t. if answer on (classical) input x is:

- Yes: ∃ poly.-sized quantum witness $|w\rangle \in \mathbb{C}^{poly(n)}$ s.t. $\Pr(U$ outputs "1" on input $|x\rangle|w\rangle) \geq \frac{2}{3}$.
- No: ∀ states $|w\rangle$, $\Pr(U$ outputs "1" on input $|x\rangle|w\rangle) \leq \frac{1}{3}$.

e.g.

- $1D$ gapped LHP ∈ NP (P)
- Factoring ∈ NP
- k-SAT$(k \geq 3)$ is NP-complete
- Factoring ∈ QMA
- k-LHP$(k \geq 2)$ is QMA-complete
- $1D$ LHP on qudits $(d \geq 8)$ is QMA-complete

Local Hamiltonian and LHP

"k-local" (quantum information) = "k-body" (physics)

k-local Hamiltonian Given $H = \sum_i h^{(i)} \in (\mathbb{C}^d)^{\otimes n}$, we say that H is a k-local Hamiltonian (or H is k-local) if $\forall$ i, $h^{(i)}$ is k-local, each interaction involving at most k particles, where $h = h_s \otimes \mathbb{I}_{[n]/S}$ and $S \subset [n]$.

- h acts non-trivially on subset s of the particles, $|s| = k$ if k-local.
- In general, no requirement that local interactions are geometrically local.

k-local Hamiltonian problem

Input: k-local Hamiltonian H on n-qudits with m local terms.

Promise: where $\lambda_0(H) \leq a$ or $\lambda_0(H) \geq b$ with $b - a \geq \frac{1}{poly(n)}$.

Output: Yes if $\lambda_0(H) \leq a$; No if $\lambda_0(H) \geq b$.

- Input is classical description of H, i.e. d^{2k} elements for each m terms.
- w.l.o.g. Input size $m \leq C_n^k = O(n^k) = poly(n)$. So matrix entries are restricted to $poly(n)$ digit of precision.

Local Hamiltonian: Examples

Transverse Ising model 2-local ($d = 2$), gapped, frustrated

$$H = \frac{1}{2\sqrt{1+h^2}} \left[\sum_{i=1}^{N-1} S_i^x S_{i+1}^x + h \sum_{i=1}^{N} S_i^z \right]$$

where $h = 1.1$ and gap $\Delta \approx 0.07$.

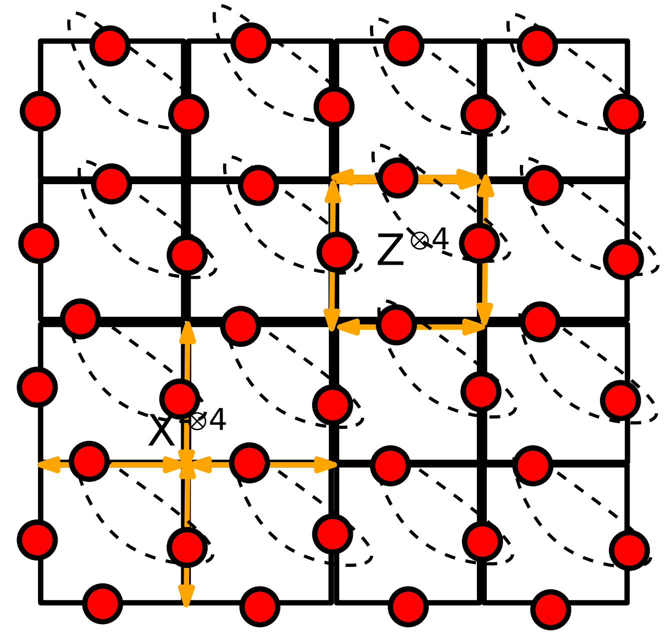
$2D$ **AKLT model on square lattice** 2-local ($d=5$), gapped, frustration-free

$$H_{AKLT}^{s=2} = \frac{1}{14} \sum_{i,j} \left[S_i \cdot S_j + \frac{7}{10}(S_i \cdot S_j)^2 + \frac{7}{45}(S_i \cdot S_j)^3 + \frac{1}{90}(S_i \cdot S_j)^4 \right]$$

Toric code 4-local ($d = 2$) or 3-local ($d = 4$), commute, frustration-free

$$H = -\sum_{v \in V} A(v) - \sum_{p \in P} B(p)$$

where vertex operators $A = X^{\otimes 4}$ and plaquette operators $B = Z^{\otimes 4}$.



Complexity class: #P

#P (the number of P) A function $f : \{0,1\}^n \to \mathbb{N} \in \#\mathsf{P}$ if $\exists$ polynomial $p : \mathbb{N} \to \mathbb{N}$ and a poly-sized uniform classical verifier V s.t. $\forall x \in \{0,1\}^n$:

$$f(x) = |\left\{ y \in \{0,1\}^{p(|x|)} : V(x,w) = 1 \right\}|$$

where w is poly-sized witness.

e.g.

- Permanent is #P-complete.

$$perm(A) = \sum_{\sigma \in S_n} \prod_{c=1}^{n} A_{i,\sigma(i)} \qquad det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

- Partition function of classical Ising model is #P-complete.

  Consider n sites system with configuration defined by $\sigma \in \{0,1\}$ on each site. Energy of a configuration σ is given by

$$H(\sigma) = -\sum_{i,j} V_{ij}\sigma_i\sigma_j - B\sum_k \sigma_k$$

  where V_{ij} is interaction energies and B is magnetic field intensity. And the partition function of the system:

$$Z = \sum_{\sigma \in \{0,1\}^n} exp(-\beta H(\sigma))$$

  where β is the inverse temperature.

Reduction

How to compare difficulty of different computational problems?

Poly-time reduction (Karp reduction) A reduces to B if $\exists$ map $A \to B$ with instance $a \mapsto$ instance b s.t. b has answer Yes iff a has answer Yes. Also the map $A \to B$ can be computed by poly-size circuit. Note A reduces to B as $A \leq B$.

Hardness Problem B is NP-hard if $\forall A \in \mathsf{NP}$, $A \leq B$.

Completeness Problem A is NP-complete if $A \in \mathsf{NP}$-hard and $A \in \mathsf{NP}$.

- "hardest problems in NP": if you can solve one, you can solve **all** NP problems.
- It can be used as the definition of complexity classes, such as all equivalent models of quantum computation.

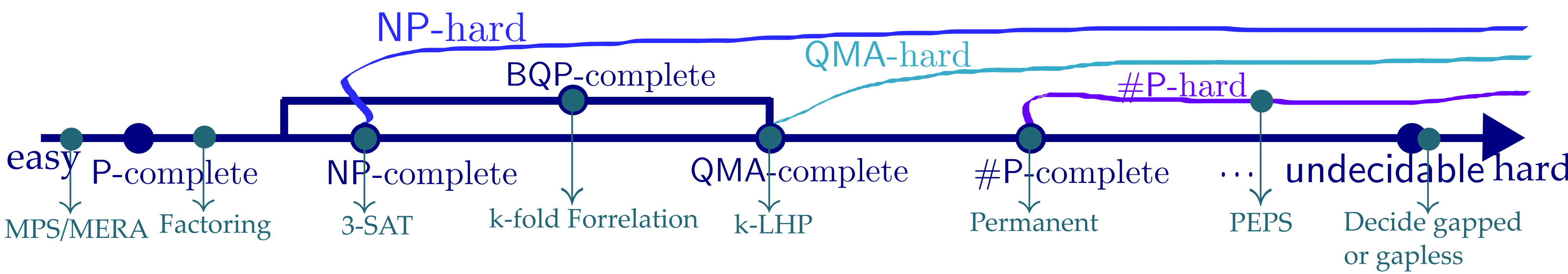
e.g.

- NP-complete: 3-SAT
- QMA-complete: k-LHP$(k \geq 2)$, $1D$ LHP on qudit $(d \geq 8)$
- #P-complete: Permanent, classical Ising model's partition function
- #P-hard: exactly contract PEPS [Schuch&Wolf&Verstraete&Cirac '06]

Complexity Zoo

Separate or Collapse? $\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{QMA} \subseteq \#\mathsf{P}$
$\mathsf{P} \subseteq \mathsf{BQP} \subseteq \mathsf{QMA} \subseteq \#\mathsf{P}$

- **Separate** $\mathsf{P} \subset \mathsf{BQP} \subset \mathsf{NP} \subset \mathsf{QMA} \subset \#\mathsf{P}$
- **Collapse** $\mathsf{P} = \mathsf{BQP} = \mathsf{NP} = \mathsf{QMA} = \#\mathsf{P}$



How to interpret these relations?

$\mathsf{P} \subseteq \mathsf{BQP}$, $\mathsf{NP} \subseteq \mathsf{QMA}$	classical = special case of quantum
$\mathsf{P} \stackrel{?}{=} \mathsf{NP}$	\$1,000,000
$\mathsf{P} \stackrel{?}{=} \mathsf{BQP}$	Are quantum computers useful?
$\mathsf{BQP} \stackrel{?}{=} \mathsf{QMA}$	quantum "P-v.s.-NP"
$\mathsf{NP} \stackrel{?}{=} \mathsf{BQP}$	Are quantum computers such powerful?

How hard is the local Hamiltonian?

Quantum Cook-Levin theorem

[Kitaev '99] k-LHP is QMA-complete ($k \geq 5$).

[Kempe&Kitaev&Regev '05] 2-LHP is QMA-complete.

- Even for $1D$ system in general, $1D$ LHP on qudit($d \geq 8$) is QMA-complete.
- Even for frustration-free systems, 3-LHP is QMA_1-complete (with perfect completeness).

Sometimes it is easier...

Commute (for local terms in Hamiltonian)

- [Bravyi&Vyalyi '06] 2-CLH on qudit is in NP
- [Aharonov&Eldar '13] 3-CLH on qubit is in NP
- [Schuch '11] 4-CLH on the square lattice is in NP
- Higher dimension of lattice or higher physical dimension ?

Gapped (Hamiltonian with spectral gap > 0)

- [Landau&Vazirani&Vidick '15] $1D$ gapped LHP is in P
- $2D$ gapped LHP is in NP (?)
- It connects to area law and tensor network.

An approach to show a class of LHP is in NP

g.s. $|\Omega\rangle$ admits on efficient classical description
$\Rightarrow$ a class of LHP is inside NP(P)

1. $|\Omega_c\rangle$ is described by $poly(N)$ classical bits.
2. $\langle\Omega_c|A|\Omega_c\rangle$ can be efficiently approximated up to $\|A\|/poly(N)$ for every local observable A.
3. $|\langle\Omega_c|A|\Omega_c\rangle - \langle\Omega|A|\Omega\rangle| \leq \|A\|/poly(N)$

$|\Omega_c\rangle$ as a classical witness for LHP, since

$$\langle\Omega|H|\Omega\rangle = \sum_X \langle\Omega|h_X|\Omega\rangle \approx \sum_X \langle\Omega_c|h_X|\Omega_c\rangle = \langle\Omega_c|H|\Omega_c\rangle.$$

local operator

1D gapped LHP is inside P(NP)

$1D$ area law

$S(\Omega) \leq \text{const}$
[Hastings '07]

Efficient description approximated up to $\|A\|/poly(N)$

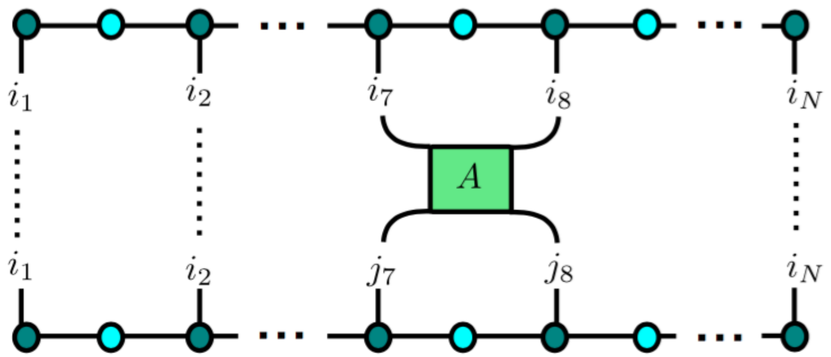
Truncate Schmidt coefficients after $poly(N)$ to get a $1/poly(N)$-approximation of $|\Omega\rangle$
$\Rightarrow$ polynomial MPS [Vidal '03]

Finding ground states

DMRG [White '92]
iTEBD [Vidal '03]
DP [LVV '15] (P)

Efficiently calculate the expectation value

Guarantee by MPS itself
[Vidal '03]



Is 2D gapped LHP inside NP?

Area law is not enough
[Ge&Eisert '15]

???

$2D$ area law

Proof is missing...

Since PEPS contraction is #P-hard in general, PEPS is not enough
[Schuch&Wolf&Verstraete&Cirac '06]

Finding ground states

iPEPS or ...
[Jordan&Orus&Vidal&Verstraete&Cirac '08]

Efficient description approximated up to $\|A\|/poly(N)$ for PEPS

decay of correlations

Strong PEPS conjecture [Schwarz&Buerschaper&Eisert '16]

For all local observables O_X and any constant $\epsilon > 0$,there exists an injective PEPS w with bond dimension $D = O(poly(N), \epsilon^{-1})$ such that its parent Hamiltonian H_* has a constant spectral gap $\Delta_* > 0$ and $|\mathrm{Tr}(O_x\rho) - \mathrm{Tr}(O_x w)| < \epsilon$.

Efficiently calculate the expectation value

Quasi-poly time algorithm [Schwarz&Buerschaper&Eisert '16]

Consider the local patch with correlation length $\log(N)$ ($N = \#$ spins) for translation-invariant system. Quasi-polynomial time algorithm $(Dd)^{O(l^d)}$, where D is the bond dimension, d is the phsycial dimension and correlation length $l = O(\log(N))$.

Reference & Further Reading

- *Quantum Computation and Complexity Course* on 2016 Autrans summer school. Toby Cubitt.
  (introductory lecture notes)
- *Classical and Quantum Computation.* Alexei Kitaev, Alexander Shen, Mikhail Vyalyi. 2002.
  (a little technical textbook)
- *Computational Complexity: A Modern Approach.* Sanjeev Arora, Boaz Barak. 2009.
  (technical textbook and a dictionary)
- Chapter 4 in the *Quantum Information Meets Quantum Matter.* Bei Zeng, Xie Chen, Duan-Lu Zhou, Xiao-Gang Wen. 2016.
  (introductory materials)
- *Quantum Hamiltonian Complexity.* QIP 2015 Tutorial. Itai Arad.
  (complexity theory perspective)
- *Matrix Product States and Tensor Network States.* QIP 2017 Tutorial. Norbert Schuch
  (tensor network perspective)
- *Quantum Hamiltonian Complexity.* Sevag Gharibion ,Yichen Huang, Zeph Landau, Seung Woo-Shin.

Q&A