

On estimating the quantum ℓ_α distance

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- 1 Quantum state testing with respect to the quantum ℓ_α distance
- 2 Main results: Upper bounds, lower bounds, and complexity classes
- 3 Proof techniques
- 4 Open problems

What is quantum state testing

Task: Closeness testing of quantum states

Given two state-preparation circuits Q_0 and Q_1 (“quantum devices”) that prepare (the purification of) n -qubit quantum states $\rho_0 \in \mathbb{C}^{N \times N}$ and $\rho_1 \in \mathbb{C}^{N \times N}$, respectively. Decide whether $\text{dist}(\rho_0, \rho_1) \geq a(n)$ or $\text{dist}(\rho_0, \rho_1) \leq b(n)$.

- ▶ Classical counterpart (distribution testing) involves probability distributions D_0 and D_1 that are samplable via respective Boolean circuits C_0 and C_1 . The task is to decide whether $\text{dist}(D_0, D_1) \geq a(n)$ or $\text{dist}(D_0, D_1) \leq b(n)$.
- ▶ Quantum devices Q_b for $b \in \{0, 1\}$ can be given either as a query oracle (*black-box model*) or a sequence of $\text{poly}(n)$ elementary quantum gates (*white-box model*).

Typical goal. Minimize the “complexity” of ρ_b (or its corresponding Q_b) for $b \in \{0, 1\}$:

Type of query access	Complexity measure
Black-box model	Query complexity (the number of queries)
White-box model	Complexity class

Generalizing the closeness measures via the Schatten α -norm

The most canonical choices of closeness measures are:

- ◇ **Trace distance** $T(\rho_0, \rho_1) = \frac{1}{2} \text{Tr}(|\rho_0 - \rho_1|)$.
- ◇ **Total variation distance** $\text{TV}(D_0, D_1) = \frac{1}{2} \sum_x |D_0(x) - D_1(x)|$.

Generalization. Define the quantum ℓ_α distance via the Schatten norm:

$$T_\alpha(\rho_0, \rho_1) := \frac{1}{2} \|\rho_0 - \rho_1\|_\alpha = \frac{1}{2} \text{Tr}(|\rho_0 - \rho_1|^\alpha)^{1/\alpha}.$$

Trace distance ($\alpha = 1$). The closeness testing problem in this case is *hard*, with complexity (polynomially) depending on the rank r of the quantum states:

- The query complexity for estimating $T(\rho_0, \rho_1)$ to within additive error ε is $\tilde{O}(r/\varepsilon^2)$ [Wang-Zhang'23] and $\tilde{\Omega}(r^{1/2})$ [Bun-Kothari-Thaler'17].
- The promise problem QUANTUM STATE DISTINGUISHABILITY ($\text{QSD}[a, b]$) is QSZK-complete* [Watrous'02, Watrous'05], and it is widely believed that $\text{BQP} \subsetneq \text{QSZK}$.
 - ◇ The QSZK containment holds only in the *polarizing regime* $a(n)^2 - b(n) > 1/O(\log n)$, rather than the *natural regime* $a(n) - b(n) \geq 1/\text{poly}(n)$.
 - ◇ The QSZK containment has recently been slightly improved beyond that in [L.'23], while the result is weaker than the classical case [Berman-Degwekar-Rothblum-Vasudevan'19].

Generalizing the closeness measures via the Schatten α -norm (Cont.)

Even $\alpha \in \{2, 4, \dots\}$. The closeness testing problem in this case is *easy*, with complexity *independent* of the rank r of the quantum states:

- ▶ The query complexity for estimating $\text{Tr}(\rho_0 \rho_1)$ to within additive error ε is $O(1/\varepsilon)$ via the SWAP test [Buhrman-Cleve-Watrous-de Wolf'01].
 - ◊ This directly applies to the case $\alpha = 2$, since $\text{Tr}((\rho_0 - \rho_1)^2) = \text{Tr}(\rho_0^2) + \text{Tr}(\rho_1^2) - 2\text{Tr}(\rho_0 \rho_1)$.
 - ◊ In the white-box model, the corresponding closeness testing problem is in BQP.
- ▶ Similar techniques [Ekert-Alves-Oi-Horodecki-Horodecki-Lwek'02] can estimate $\text{Tr}(\rho_1 \rho_2 \cdots \rho_k)$ for integer $k > 1$, and solve the case of even integers α .

Odd $\alpha = 3$. The query complexity of deciding whether $T_3(\rho_0, \rho_1) = \frac{1}{2} \text{Tr}(|\rho_0 - \rho_1|^3)^{1/3} \geq \varepsilon$ or $\rho_0 = \rho_1$ is $O(1/\varepsilon^{3/2})$ [Gilyén-Li'19], but **this does not extend to the estimation task**:

- ▶ $T_3(\rho_0, \rho_1) \geq \varepsilon \Rightarrow \text{Tr}\left(\left(\frac{\rho_0 - \rho_1}{2}\right)^2 \left(\frac{\rho_0 + \rho_1}{2}\right)\right) \geq \frac{1}{8} \text{Tr}[|\rho_0 - \rho_1|^3] \geq \varepsilon^3$.

$\alpha > 1$ in general. For real-valued $\alpha > 1$, the query complexity for estimating $T_\alpha(\rho_0, \rho_1)$ to within additive error ε is $\text{poly}(r, 1/\varepsilon)$ [Wang-Guan-Liu-Zhang-Ying'22], which polynomially depends on *the rank* of the states of interests.

Generalizing the closeness measures via the Schatten α -norm (Cont.²)

What about the complexity of estimating the classical ℓ_α distance?

Similarly, define the classical ℓ_α distance $\text{TV}_\alpha(D_0, D_1) := \frac{1}{2} (\sum_x |D_0(x) - D_1(x)|^\alpha)^{1/\alpha}$.

- ▶ For real-valued $\alpha > 1$, the sample complexity of estimating $\text{TV}_\alpha(D_0, D_1)$ to within additive error ε is $\text{poly}(1/\varepsilon)$, which is *independent of the support size* of distributions D_0 and D_1 , and fewer samples are needed as α increases [Waggoner'14].
- ▶ **Intuition:** When $\varepsilon = \Theta(1)$, draw $\text{poly}(n)$ samples from D_0 and D_1 , and compute the classical ℓ_α distance between the resulting empirical distributions.
- ▶ **Issue:** This intuition does not work in the quantum world... ☹

Hope: Estimating the trace of quantum state powers is easy for real-valued $q > 1$.

The query complexity of estimating $\text{Tr}(\rho^q)$ for real-valued $q > 1$ is $\text{poly}(1/\varepsilon)$ [L.-Wang'24].

- ▶ **Issue:** Since $|\rho_0 - \rho_1|$ is *not* a quantum state, this does not apply to $T_\alpha(\rho_0, \rho_1)$. ☹

Question: What is the complexity of estimating $T_\alpha(\rho_0, \rho_1)$ for real-valued $\alpha > 1$?

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Main results: Upper bounds

Theorem 1 (Quantum estimator for quantum ℓ_α distance).

Given quantum query access to the state-preparation circuit Q_0 and Q_1 for the n -qubit state ρ_0 and ρ_1 , for any real-valued $\alpha > 1$, there is a quantum algorithm that estimates $T_\alpha(\rho_0, \rho_1)$ to within additive error ε , with query complexity $O(1/\varepsilon^{\alpha+1+\frac{1}{\alpha-1}}) = \text{poly}(1/\varepsilon)$.

- ▶ The corresponding closeness testing problem $\text{QSD}_\alpha[a(n), b(n)]$ decides whether $T_\alpha(\rho_0, \rho_1)$ is at least $a(n)$ or at most $b(n)$, e.g., $a(n) = 2/5$ and $b(n) = 1/5$.
 - ▶ As a corollary, for any real-valued $\alpha > 1$ and all $a(n), b(n) \in [0, 1]$ satisfying $a(n) - b(n) \geq 1/\text{poly}(n)$, $\text{QSD}_\alpha[a(n), b(n)]$ is in BQP.
 - ▶ Using the (multi-)sampler [Wang-Zhang'23, Wang-Zhang'24], which enables a quantum query-to-sample simulation, the sample complexity used to estimate $T_\alpha(\rho_0, \rho_1)$ is $\tilde{O}(1/\varepsilon^{3\alpha+2+\frac{2}{\alpha-1}}) = \text{poly}(1/\varepsilon)$.
- 📌 While the prior best result [Wang-Guan-Liu-Zhang-Ying'22] has complexity polynomially depending on the rank r of ρ_0 and ρ_1 , **our work is rank-independent and thus provides an exponential improvement!**

Main results: Complexity classes & lower bounds

Let PUREQSD_α be a restricted variant of QSD_α , where the states of interest are *pure*:

Theorem 2 (Computational hardness of QSD_α).

The promise problem QSD_α captures the computational power of the respective complexity classes, depending on the regime of α :

- ① **Easy regimes.** For any $1 \leq \alpha \leq \infty$, PUREQSD_α (with constant precision) is BQP-hard. Consequently, for real-valued $\alpha > 1$, QSD_α is BQP-complete.
- ② **Hard regimes.** For any $\alpha \in (1, 1 + \frac{1}{n}]$, QSD_α is QSZK-complete*.
 - The QSZK containment of $\text{QSD}_\alpha[a, b]$ holds only in the polarizing regime $a(n)^2 - b(n) \geq 1/O(\log n)$.

📌 A sharp phase transition occurs between the case of $\alpha = 1$ and real-valued $\alpha > 1$!

Our reductions used to establish the hardness also leads to quantitative (query & sample complexity) lower bounds for estimating $T_\alpha(\rho_0, \rho_1)$ to within additive error ε :

The regime of α	$1 < \alpha \leq 1 + \frac{1}{n^{1+\delta}}$	$1 + \frac{1}{n^{1+\delta}} < \alpha \leq 1 + \frac{1}{n}$	Real-valued $\alpha > 1$
Query complexity	$\tilde{\Omega}(r^{1/2})$	$\Omega(r^{1/3})$	$\Omega(1/\varepsilon)$
Sample complexity	$\Omega(r/\varepsilon^2)$		$\Omega(1/\varepsilon^2)$

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Proof techniques: BQP containment of QSD $_{\alpha}$ for real-valued $\alpha > 1$

We begin by reviewing the approach in [Wang-Zhang'23] for estimating the trace distance ($\alpha = 1$), which uses the following key identity to decompose $T(\rho_0, \rho_1)$:

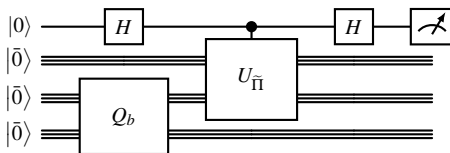
$$T(\rho_0, \rho_1) = \frac{1}{2} \text{Tr} \left(\rho_0 \text{sgn} \left(\frac{\rho_0 - \rho_1}{2} \right) \right) - \frac{1}{2} \text{Tr} \left(\rho_1 \text{sgn} \left(\frac{\rho_0 - \rho_1}{2} \right) \right) = \text{Tr}(\Pi \rho_0) - \text{Tr}(\Pi \rho_1).$$

The Holevo-Helstrom measurement $\{\Pi, I - \Pi\}$ and its approx. implementation satisfy:

$$\Pi = \frac{I}{2} + \frac{1}{2} \text{sgn} \left(\frac{\rho_0 - \rho_1}{2} \right) \quad \text{and} \quad \tilde{\Pi} = \frac{I}{2} + \frac{1}{2} P_d^{\text{sgn}} \left(\frac{\rho_0 - \rho_1}{2} \right).$$

Implementing Π approximately. Using Quantum Singular Value Transformation

[Gilyén-Su-Low-Wiebe'19] with a “good” polynomial approximation $P_d^{\text{sgn}}(x)$ of the sign function $\text{sgn}(x)$ on the interval $[-1, 1] \setminus (-\delta, \delta)$, with degree $d = O(\frac{1}{\delta} \log \frac{1}{\epsilon})$, one can approx. implement the HH measurement via the Hadamard test [Kitaev'95, Aharonov-Jones-Landau'06]:



- **Algorithmic HH measurement** [Le Gall-L.-Wang'23]: Using the *space-efficient* QSVT, for any n -qubit quantum states ρ_0 and ρ_1 , $\tilde{\Pi}$ can be implemented in $\text{poly}(N)$ time and $O(n)$ space, where $N = 2^n$ is the dimension of the states.

BQP containment of QSD_α for real-valued $\alpha > 1$ (Cont.)

Inspired by the identity ($\alpha = 1$) used in [Wang-Zhang'23], we use the following identity to decompose the *powered* quantum ℓ_α distance $\Lambda_\alpha(\rho_0, \rho_1) = 2^{\alpha-1} \text{Tr}(\rho_0, \rho_1)^\alpha$:

$$\begin{aligned}\Lambda_\alpha(\rho_0, \rho_1) &:= \frac{1}{2} \text{Tr}(|\rho_0 - \rho_1|^\alpha) = \frac{1}{2} \text{Tr}(\rho_0 \cdot \text{sgn}(v) |v|^{\alpha-1}) - \frac{1}{2} \text{Tr}(\rho_1 \cdot \text{sgn}(v) |v|^{\alpha-1}) \\ &= \text{Tr}(\Pi_\alpha \rho_0) - \text{Tr}(\Pi_\alpha \rho_1), \\ \text{where } v &= \rho_0 - \rho_1 \text{ and } \Pi_\alpha := \frac{I}{2} + \frac{1}{2} \text{sgn}(v) |v|^{\alpha-1}.\end{aligned}$$

Similar to the case $\alpha = 1$, we can approximately implement Π_α via QSVT and the Hadamard test, denoted as $\tilde{\Pi}_\alpha$, using an *approximate* polynomial approximation $P_d(x)$ of the function $\text{sgn}(x)|x|^\beta$, where $\beta = \alpha - 1 > 0$ is a real number.

Removing the rank dependence. Unlike the case $\alpha = 1$, we need a polynomial $P_d(x)$ that *uniformly* approximate $\text{sgn}(x)|x|^\beta$. The best uniform (polynomial) approximation of x^β was originally investigated in [Bernstein'38], and the signed version $\text{sgn}(x)|x|^\beta$ was listed in [Totik'06] and a non-constructive proof is provided in [Ganzburg'08]:

$$\max_{x \in [-1, 1]} \left| P_{d^*}^*(x) - \text{sgn}(x)|x|^\beta \right| \rightarrow (1/d^*)^\beta, \quad \text{as } d^* \rightarrow \infty.$$

By the Chebyshev truncation and the de La Vallée Poussin partial sum, we can make the coefficients of $P_{d^*}^*(x)$ *efficiently computable* with a slightly larger degree $d = 2d^* - 1$:

$$\max_{x \in [-1, 1]} \left| P_d(x) - \frac{1}{2} \text{sgn}(x)|x|^\beta \right| \leq \varepsilon \quad \text{and} \quad \max_{x \in [-1, 1]} |P_d(x)| \leq 1, \quad \text{where } d = O(1/\varepsilon^{1/\beta}).$$

Proof techniques: Lower bounds via a new inequality between T and T_α

By carefully analyzing the properties of *orthogonal* PSD matrices ζ_0 and ζ_1 such that $\rho_0 - \rho_1 = \zeta_0 - \zeta_1$, we establish a new *rank-dependent* inequality between T and T_α :

Theorem 3 (T vs. T_α). For any quantum states ρ_0 and ρ_1 ,

$$\forall \alpha \in [1, \infty], \quad 2^{1-\frac{1}{\alpha}} \cdot T_\alpha(\rho_0, \rho_1) \leq T(\rho_0, \rho_1) \leq 2(\text{rank}(\rho_0)^{1-\alpha} + \text{rank}(\rho_1)^{1-\alpha})^{-\frac{1}{\alpha}} \cdot T_\alpha(\rho_0, \rho_1).$$

- ▶ The case of $\alpha = 2$ was previously proven in [Coles'11, Coles-Cerezo-Cincio'19].
- ▶ The inequalities in Theorem 3 are *sharper* than those between the trace norm and the Schatten norm, such as in [Aubrun-Szarek'17]:

$$\forall \alpha \in [1, \infty], \quad \|A\|_\alpha \leq \|A\|_1 \leq \text{rank}(A)^{1-\frac{1}{\alpha}} \|A\|_\alpha.$$

- ▶ For pure states $|\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle\langle\psi_1|$, the following identity holds:

$$\forall \alpha \in [1, \infty], \quad 2^{1-\frac{1}{\alpha}} \cdot T_\alpha(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|) = T(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|).$$

Reductions via inequalities in Theorem 3. Consequently, we obtain:

- ▶ Reductions from the case $\alpha = 1$ (e.g., QSD) to the case $\alpha > 1$ (e.g., QSD_α), with the relevant $\alpha > 1$ ranges differing between QSD_α and PUREQSD_α .
- ▶ This implies that the computational hardness and lower bounds for QSD_α and PUREQSD_α follow from the prior works on the trace distance ($\alpha = 1$).

Proof techniques: QSZK containment of QSD_α for $1 < \alpha < 1 + \frac{1}{n}$

Directly combining the inequalities in Theorem 3 and the QSZK containment of QSD in [Watrous'02, Watrous'05] implies a QSZK containment of $\text{QSD}_\alpha[a, b]$ under a parameter regime $a(n)^2/2 - b(n) \geq 1/O(\log n)$, which is **even worse than the polarizing regime**. ☹

Lemma 4 (A *partial* polarization lemma for T_α). Given a QSD_α instance (Q_0, Q_1, a, b, k) , there exists a deterministic procedure that outputs new quantum circuits \tilde{Q}_0 and \tilde{Q}_1 that prepare the states $\tilde{\rho}_0$ and $\tilde{\rho}_1$. The resulting states satisfy the following:

$$\begin{aligned} \forall \alpha \in \left[1, 1 + \frac{1}{n}\right], \quad T_\alpha(\rho_0, \rho_1) \geq a &\implies T_\alpha(\tilde{\rho}_0, \tilde{\rho}_1) \geq (1 - e^{-k})/2, \\ T_\alpha(\rho_0, \rho_1) \leq b &\implies T_\alpha(\tilde{\rho}_0, \tilde{\rho}_1) \leq 1/16. \end{aligned}$$

When $k \leq O(1)$ or $a^2 - b \geq \Omega(1)$, the time complexity of the procedure is polynomial in the size of Q_0 and Q_1 , k , and $\exp\left(\frac{b \log(1/a^2)}{a^2 - b}\right)$.

- ▶ The polarization lemma for T makes $T(\tilde{\rho}_0, \tilde{\rho}_1)$ either at least $1 - 2^{-k}$ or at most 2^{-k} :
 - ◇ The difference is because the direct product lemma for $1 < \alpha < 1 + \frac{1}{n}$ is weaker than $\alpha = 1$;
 - ◇ I.e., $T_\alpha(\rho_0^{\otimes l}, \rho_1^{\otimes l}) \geq \frac{1}{2} - \frac{1}{2} \exp\left(-\frac{l}{2} \cdot T_\alpha(\rho_0, \rho_1)^2\right)$ vs. $T(\rho_0^{\otimes l}, \rho_1^{\otimes l}) \geq 1 - \exp\left(-\frac{l}{2} \cdot T(\rho_0, \rho_1)^2\right)$.
- ▶ The QSZK containment of QSD_α for $1 < \alpha \leq 1 + \frac{1}{n}$ is the following:
 - ① Apply the partial polarization lemma for T_α (Lemma 4) to the QSD_α instance;
 - ② Combine with the inequalities in Theorem 3 to produce a QSD instance;
 - ③ Use the QSZK containment of QSD in [Watrous'02, Watrous'05].

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Conclusions and open problems

Take-home messages on our work

- 1 For the regime $\alpha \geq 1 + \Omega(1)$, estimating the quantum ℓ_α distance $T_\alpha(\rho_0, \rho_1)$ is computationally *easy* and has *rank-independent* query & sample complexities.
- 2 For the regime $1 < \alpha \leq 1 + \frac{1}{n}$, estimating the quantum ℓ_α distance $T_\alpha(\rho_0, \rho_1)$ is computationally *hard* and the query & sample complexities are *rank-dependent*.

Discussion and open problems

While $T_\alpha(\rho_0, \rho_1)$ and its powered version $\Lambda_\alpha(\rho_0, \rho_1)$ are *almost interchangeable* for real-valued $\alpha > 1$, their behavior differs significantly when $\alpha = \infty$:

- ▶ $T_\infty(\rho_0, \rho_1)$ corresponds to the largest eigenvalue $\lambda_{\max}(\frac{\rho_0 - \rho_1}{2})$.
 - ◇ Estimating $\text{TV}_\infty(D_0, D_1)$ to within additive error ε uses $O(1/\varepsilon^2)$ samples [Waggoner'14].
 - ◇ The pure state version PUREQSD_∞ is BQP-complete, while we only know that the general version QSD_∞ is contained in QMA.
- ▶ $\Lambda_\infty(\rho_0, \rho_1) \in \{0, \frac{1}{2}, 1\}$ for any quantum states ρ_0, ρ_1 , and it is nonzero if and only if the states are orthogonal with at least one of them being pure.
 - ▶ The pure state version $\text{PUREPOWEREDQSD}_\infty[1, 0]$ is coNQP-hard, which is a precise variant of BQP that ensures acceptance for all *yes* instances.

Question: What is the computational complexity of estimating $T_\infty(\rho_0, \rho_1)$?

Thanks!