Space-bounded quantum state testing

via space-efficient quantum singular value transformation

François Le Gall, **Yupan Liu**, Qisheng Wang

Nagoya University

Available at arXiv:2308.05079

Bristol Quantum Information Theory Seminar, Mar 2024

What is quantum state testing

Task: Quantum state testing (with two-sided error).

Given two quantum devices Q_0 and Q_1 that prepare $\mathrm{poly}(n)$-qubit quantum (mixed) states $\rho_0 \in \mathbb{C}^{N\times N}$ and $\rho_1 \in \mathbb{C}^{N\times N}$, respectively, which may be viewed as "sample access" to ρ_0 and ρ_1. Decide whether $\mathrm{dist}(\rho_0,\rho_1) \leq \epsilon_1$ or $\mathrm{dist}(\rho_0,\rho_1) \geq \epsilon_2$.

The one-sided error variant and the classical counterpart are as follows:

- **Quantum state certification** [Bădescu-O'Donnell-Wright'19]:
  Given "sample access" to ρ_0 and ρ_1, decide whether $\rho_0 = \rho_1$ or $\mathrm{dist}(\rho_0,\rho_1) \geq \epsilon$.
- **Distribution testing** (a.k.a. closeness testing of distributions, see [Canonne'20]):
  Given sample accesses to probability distributions D_0 and D_1, decide whether $\mathrm{dist}(D_0,D_1) \leq \epsilon_1$ or $\mathrm{dist}(D_0,D_1) \geq \epsilon_2$.

Typical goal: Minimize the number of copies (*sample complexity*) of ρ_0 and ρ_1.

In this work: Viewing quantum state testing as a computational (promise) problem.

Quantum state testing w.r.t. various distance-like measures

Classical and quantum distance-like measures that are considered:

	Quantum	Classical
ℓ_1 norm	trace distance $\mathrm{td}(\rho_0,\rho_1) := \frac{1}{2}\mathrm{Tr}\lvert\rho_0-\rho_1\rvert$	total variation distance (a.k.a. statistical distance)
ℓ_2 norm	Hilbert-Schmidt distance $\mathrm{HS}^2(\rho_0,\rho_1) := \frac{1}{2}\mathrm{Tr}(\rho_0-\rho_1)^2$	Euclidean distance
Entropy	von Neumann entropy $S(\rho) := -\mathrm{Tr}(\rho\ln\rho)$	Shannon entropy
Jensen-Shannon divergence	Quantum Jensen-Shannon divergence $\mathrm{QJS}_2(\rho_0,\rho_1) := S_2\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{S_2(\rho_0)+S_2(\rho_1)}{2}$ where $S_2(\rho) := -\mathrm{Tr}(\rho\log_2\rho)$ [Majtey-Lamberti-Prato'05, Briët-Harremoës'09]	Jensen-Shannon divergence

Remark. Quantum Jensen-Shannon divergence can be viewed as a *distance version* of the quantum (von Neumann) entropy difference.

Main result: Space-bounded state certification (one-sided error scenario)

Task 1.1 (Space-bounded quantum state certification). Given two *polynomial-size* $O(\log n)$-*qubit* quantum circuits Q_0 and Q_1 that prepare $O(\log n)$-qubit quantum (mixed) states ρ_0 and ρ_1, respectively, with access to their "source codes". Decide whether $\rho_0 = \rho_1$ or $\mathrm{dist}(\rho_0, \rho_1) \geq \alpha$.

Theorem 1.2 (Space-bounded quantum state certification is $\mathsf{coRQ_UL}$-complete). The following space-bounded quantum state certification problems are $\mathsf{coRQ_UL}$-complete. For any $\alpha(n) \geq 1/\mathrm{poly}(n)$, decide whether

1. $\overline{\textsc{CertQSD}}_{\mathsf{log}}$: $\rho_0 = \rho_1$ or $\mathrm{td}(\rho_0, \rho_1) \geq \alpha(n)$;
2. $\overline{\textsc{CertQHS}}_{\mathsf{log}}$: $\rho_0 = \rho_1$ or $\mathrm{HS}^2(\rho_0, \rho_1) \geq \alpha(n)$.

Remark. $\mathsf{coRQ_UL}$ is a complexity class with *perfect completeness*, namely the acceptance probability $p_{\mathrm{acc}} = 1$ for *yes* instances whereas $p_{\mathrm{acc}} \leq 1/2$ for *no* instances.

Main result: Space-bounded quantum state testing (two-sided error scenario)

Task 1.3 (Space-bounded quantum state testing). Given two *polynomial-size* $O(\log n)$*-qubit* quantum circuits Q_0 and Q_1 that prepare $O(\log n)$-qubit quantum (mixed) states ρ_0 and ρ_1, respectively, with access to their "source codes". Decide whether $\mathrm{dist}(\rho_0, \rho_1) \leq \beta$ or $\mathrm{dist}(\rho_0, \rho_1) \geq \alpha$.

Theorem 1.4 (Space-bounded quantum state testing is BQL-complete). The following space-bounded quantum state testing problems are BQL-complete. For any α, β such that $\alpha(n) - \beta(n) \geq 1/\mathrm{poly}(n)$ or any $g(n) \geq 1/\mathrm{poly}(n)$, decide whether

1. $\textsc{GapQSD}_{\mathsf{log}}$: $\mathrm{td}(\rho_0, \rho_1) \geq \alpha$ or $\mathrm{td}(\rho_0, \rho_1) \leq \beta$;
2. $\textsc{GapQHS}_{\mathsf{log}}$: $\mathrm{HS}^2(\rho_0, \rho_1) \geq \alpha$ or $\mathrm{HS}^2(\rho_0, \rho_1) \leq \beta$;
3. $\textsc{GapQED}_{\mathsf{log}}$: $S(\rho_0) - S(\rho_1) \geq g$ or $S(\rho_1) - S(\rho_0) \geq g$;
4. $\textsc{GapQJS}_{\mathsf{log}}$: $\mathrm{QJS}_2(\rho_0, \rho_1) \geq \alpha$ or $\mathrm{QJS}_2(\rho_0, \rho_1) \leq \beta$.

BQL and $\mathsf{BQ_UL}$: Two-sided error space-bounded quantum computation

BQL (and $\mathsf{BQ_UL}$ if only allow *unitary* gates), defined in [Watrous'99, Watrous'03], captures quantum computation that performs by a *logspace-uniform* $O(\log n)$*-qubit* quantum circuit. This class admits the following properties:

- Quadratic advantage in space (?): $\mathsf{BQL} \subseteq \mathsf{DSPACE}[\log^2(n)]$ [Wat99, Wat03].
- Gateset-indep.: Space-efficient Solovay-Kitaev theorem [van Melkebeek-Watson'12].
- Error reduction for $\mathsf{BQ_UL}$ [Fefferman-Kobayashi-Lin-Morimae-Nishimura'16].
- Intermediate measurements are useless: $\mathsf{BQL} = \mathsf{BQ_UL}$ [Fefferman-Remscrim'21].

History of the only family of (natural) BQL-complete problem:

- Inverting a well-conditioned matrix is in BQL [Ta-Shma'13], whereas it only has $\mathsf{DSPACE}[\log^2(n)]$ containment without the help of quantum.
- Inverting a well-conditioned matrix is $\mathsf{BQ_UL}$-complete [Fefferman-Lin'18].
- A well-conditioned version of DET^*-complete problems are BQL-complete [Fefferman-Remscrim'21], such as well-conditioned integer determinant, well-conditioned matrix powering, well-conditioned iterative matrix product.

Takeaway. This work (Theorem 1.4) presents *a new family* of natural BQL-complete problems that emerge from quantum property testing.

$\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$: One-sided error space-bounded quantum computation

$\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$, defined in [Watrous'01], capture *one-sided error* quantum computation that performs by a *logspace-uniform* $O(\log n)$-*qubit* quantum circuits.

These classes admit the following properties:

- Error reduction for $\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$ [Watrous'01].
- Gateset-dependence which is because of perfect completeness or soundness.
- Undirected graph connectivity (USTCON) is in $\mathsf{RQ_UL} \cap \mathsf{coRQ_UL}$ [Watrous'01], although USTCON is actually in L [Reingold'08] .

Open problems on $\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$:

- A (natural) complete problem for the class $\mathsf{RQ_UL}$ or $\mathsf{coRQ_UL}$ remains *unknown*. A "verification" version of well-conditioned iterative matrix product problem is coRQL-hard [Fefferman-Remscrim'21] while there is no containment (*hard direction*).
- $\mathsf{RQ_UL} \stackrel{?}{=} \mathsf{RQL}$ and $\mathsf{coRQ_UL} \stackrel{?}{=} \mathsf{coRQL}$.

Takeaway. This work (Theorem 1.2) demonstrates *the first family* of natural $\mathsf{coRQ_UL}$-complete problems that arise from quantum property testing as well.

Time-bounded quantum state testing: ℓ_1 norm scenario

Task 3.1 (Time-bounded quantum state testing). Given two *polynomial-size* quantum circuits Q_0 and Q_1 that prepare $\mathrm{poly}(n)$-qubit quantum (mixed) states ρ_0 and ρ_1, respectively, with access to their "source codes".
Decide whether $\mathrm{dist}(\rho_0, \rho_1) \leq \beta$ or $\mathrm{dist}(\rho_0, \rho_1) \geq \alpha$.

Time-bounded distribution testing. Given two *efficiently samplable* distributions D_0 and D_1 (prepared by circuits), decide whether $\mathrm{dist}(D_0, \mathrm{D}_1) \leq \beta$ or $\mathrm{dist}(D_0, D_1) \geq \alpha$.

Computational hardness of these tasks with respect to ℓ_1 norm:

- Statistical Difference Problem (SDP) is SZK-complete
  when *constant* $\alpha^2 - \beta > 0$ [Sahai-Vadhan'03, Goldreich-Sahai-Vadhan'98].
- Quantum State Distinguishability Problem (QSDP) is QSZK-complete
  when *constant* $\alpha^2 - \beta > 0$ [Watrous'02, Watrous'09].
- Open problem: (α, β)-QSDP is in QSZK when $\alpha(n) - \beta(n) \geq 1/\mathrm{poly}(n)$.
  Recent progress: [Berman-Degwekar-Rothblum-Vasudevan'19] (classical) and [**Liu**'23] (quantum).

Structural complexity-theoretic results regarding QSZK:

- BQP $\subseteq$ QSZK $\subseteq$ QIP(2) $\subseteq$ PSPACE [Watrous'02, Watrous'09].
- $\exists \mathcal{O}$ s.t. $\mathsf{QSZK}^{\mathcal{O}} \not\subseteq \mathsf{PP}^{\mathcal{O}}$ [Bouland-Chen-Holden-Thaler-Vasudevan'19].

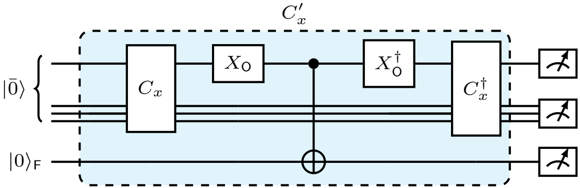
Time-bounded quantum state testing: ℓ_2 norm scenario

Proposition 3.2 [BCWdW01, RASW23]. Quantum Hilbert-Schmidt distance problem, namely time-bounded quantum state testing w.r.t. ℓ_2 norm, is BQP-complete.

BQP **containment**. As all three terms in $\mathrm{HS}^2(\rho_0,\rho_1) = \frac{1}{2}\mathrm{Tr}(\rho_0^2) + \frac{1}{2}\mathrm{Tr}(\rho_1^2) - \mathrm{Tr}(\rho_0\rho_1)$ can be estimated by the SWAP test [Buhrman-Cleve-Watrous-de Wolf'01], we have a hybrid algorithm succeeds w.p. $\frac{1}{2} + \frac{1}{2}\mathrm{HS}^2(\rho_0,\rho_1)$:

1. Toss two random coins $r_0, r_1 \in \{0,1\}$;
2. Perform the SWAP test on quantum states according to r_0 and r_1.

BQP **hardness** (adapted from [Rethinasamy-Agarwal-Sharma-Wilde'23]). Consider a BQP circuit C_x, we can construct $C'_x := C_x^\dagger X_{\mathsf{O}}^\dagger \mathrm{CNOT}_{\mathsf{O}\to\mathsf{F}} X_{\mathsf{O}} C_x$ with an ancillary qubit on F such that $\Pr[C'_x \text{ accepts}] = \|(|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_{\mathsf{F}}) C'_x (|\bar{0}\rangle \otimes |0\rangle_{\mathsf{F}})\|_2^2 = \Pr^2[C_x \text{ accepts}]$.



By defining two pure states $\rho_0 := |\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_{\mathsf{F}}$ and $\rho_1 := C'_x(|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_{\mathsf{F}})(C'_x)^\dagger$, we have $\Pr[C'_x \text{ accepts}] = \mathrm{Tr}(\rho_0\rho_1) = 1 - \mathrm{HS}^2(\rho_0,\rho_1)$. □

Computational hardness of time-bounded testing depends on distance

Computational hardness of Task 3.1: a "dichotomy" theorem

- Time-bounded state testing w.r.t. ℓ_1 norm or entropy difference (QSZK-complete) is seemingly *much harder* than only preparing these states (in BQP).
- Time-bounded state testing w.r.t. ℓ_2 norm (BQP-complete) is computationally *as easy as* just preparing these states (in BQP).

Interestingly, the computational hardness "dichotomy" is linkded to the dependence of the sample complexity for distribution testing and state testing on the *dimension* N:

	ℓ_1 norm	ℓ_2 norm	Entropy
Classical sample complexity	$\mathrm{poly}(N, 1/\epsilon)$ [CDVV14]	$\mathrm{poly}(1/\epsilon)$ [CDVV14]	$\mathrm{poly}(N, 1/\epsilon)$ [JVHW15, WY16]
Quantum sample complexity	$\mathrm{poly}(N, 1/\epsilon)$ [BOW19]	$\mathrm{poly}(1/\epsilon)$ [BOW19]	$\mathrm{poly}(N, 1/\epsilon)$ [AISW20, OW21]

Summary: Time-bounded and space-bounded testing

Computational hardness of time- and space-bounded distribution and state testing:

	ℓ_1 norm	ℓ_2 norm	Entropy
Classical Time-bounded	SZK-complete* [SV03,GSV98]	BPP-complete Folklore	SZK-complete [GV99,GSV98]
Quantum Time-bounded	QSZK-complete* [Wat02,Wat09]	BQP-complete [BCWdW01, RASW23]	QSZK-complete [BASTS10]
Classical Space-bounded	BPL-hard†	BPL-complete† Folklore	BPL-complete† Implied by [ABIS19]
Quantum Space-bounded	BQL-complete This work	BQL-complete [BCWdW01] and this work	BQL-complete This work

Remark†. Space-bounded distribution testing can be viewed as a "white-box" version of *streaming distribution testing* with i.i.d. samples.

Takeaways. For space-bounded state testing and certification problems, the computational hardness of these problems is *as easy as* just preparing quantum states, which is *independent of the choice* of aforementioned distance-like measures.

Quantum singular value transformation in a nutshell

QSVT [Gilyén-Su-Low-Wiebe'19] is a systematic approach to (time-efficiently) *manipulating singular values* $\{\sigma_i\}_i$ of an Hermitian matrix A using a corresponding projected unitary encoding $A = \tilde{\Pi} U \Pi$ for orthogonal projectors $\tilde{\Pi}$ and Π.

Quantum singular value transformation, revisited

Given a singular value decomposition $A = \sum_i \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$ associated with an $s(n)$-qubit *projected unitary encoding*, we can *approximately* implement a QSVT $f^{(\mathrm{SV})}(A) = \sum_i f(\sigma_i)|\tilde{\psi}_i\rangle\langle\psi_i|$ by employing a polynomial P_d of degree $d = O\left(\frac{1}{\delta}\log\frac{1}{\epsilon}\right)$ satisfying that

- P_d well-approximates f on the interval of interest $\mathcal{I}$: $\max_{x\in\mathcal{I}\setminus\mathcal{I}_\delta} |P_d(x) - f(x)| \leq \epsilon$ where $\mathcal{I}_\delta \subseteq \mathcal{I} \subseteq [-1,1]$ and typically $\mathcal{I}_\delta := (-\delta, \delta)$.
- P_d is bounded: $\max_{x\in[-1,1]} |P_d(x)| \leq 1$.

Moreover, all coefficients of P_d (namely, *classical pre-processing*) can be computed in deterministic $\mathrm{poly}(d)$ time (and thus space). Hence, the transformation $P_d^{(\mathrm{SV})}(A)$ can be implemented by a $\mathrm{poly}(d)$-size quantum circuit acts on $O(\max\{\log d, s(n)\})$ qubits.

Remark. Quantum circuit implementation in QSVT is already space-efficient!

Space-efficient quantum singular value transformation

Question 4.1 (Space-efficient QSVT). Can we implement a degree-d QSVT for any $s(n)$-qubit projected unitary encoding with $d \leq 2^{O(s(n))}$, using only $O(s(n))$ space in both classical pre-processing and quantum circuit implementation?

Theorem 4.2 (Space-*bounded* QSVT, [Metger-Yuen'23]). Implement a degree-d QSVT associated with *sign function* or *square-root function* for any $O(\log n)$ qubit *block-encoding* with $d \leq \mathrm{poly}(n)$ requires $O(\mathrm{poly}\log n)$ space for classical pre-processing and $O(\log n)$ qubits in quantum circuit implementation.

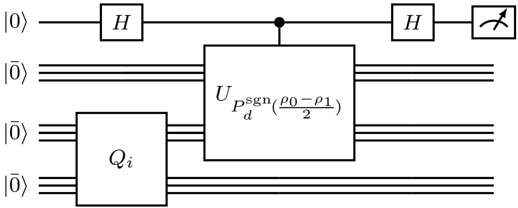
Remark. Theorem 4.2 can be easily extended to continuous functions bounded on $[-1, 1]$.

Theorem 4.3 (Space-*efficient* QSVT, This work). Implement a degree-d QSVT associated with *piecewise-smooth functions* for any $O(\log n)$ qubit *bitstring indexed encoding* with $d \leq \mathrm{poly}(n)$ requires (randomized) $O(\log n)$ space for classical pre-processing and $O(\log n)$ qubits in quantum circuit implementation.
Moreover, the implementation requires $O(d^2\|\mathbf{c}\|_1)$ uses of U, $U^\dagger$, $\mathrm{C}_\Pi\mathrm{NOT}$, $\mathrm{C}_{\tilde{\Pi}}\mathrm{NOT}$, among with other gates, where $\mathbf{c}$ is the coeffs of *Chebyshev interpolation polynomial*.

E.g. Normalized log function $\ln_\beta(x) := \frac{2\ln(1/x)}{2\ln(2/\beta)}$ on the interval $\mathcal{I} = [\beta, 1]$ for any $\beta \geq 1/\mathrm{poly}(n)$.

Proof overview: two-sided error scenario

Proof of Theorem 1.4 ①: $\textsc{GapQSD}_{\log} \in \mathsf{BQL}$. Inspired by the approach in [Gilyén-Poremba'22, Wang-Zhang'23], note that $\mathrm{sgn}(x) \approx_{\epsilon,\delta} P_d^{\mathrm{sgn}}(x)$ and $\mathrm{td}(\rho_0,\rho_1) = \frac{1}{2}\mathrm{Tr}|\rho_0-\rho_1| = \frac{1}{2}\Big(\mathrm{Tr}\Big(\mathrm{sgn}^{(\mathrm{SV})}(\frac{\rho_0-\rho_1}{2})\rho_0\Big) - \mathrm{Tr}\Big(\mathrm{sgn}^{(\mathrm{SV})}(\frac{\rho_0-\rho_1}{2})\Big)\rho_1\Big)$.
We consider the following quantum tester $\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\mathrm{sgn}})$ where $\rho_- := \frac{\rho_0-\rho_1}{2}$:



Here, Q_i prepares a purification of the state ρ_i for $i \in \{0,1\}$, and ρ_- is block-encoded in U_{ρ_-}. We say that the tester $\mathcal{T}$ accepts if the measurement outcome is "0".

By using the space-efficient QSVT (Theorem 4.3) associated with a bounded approx polynomial P_d^{sgn} of sgn, we implement $U_{P_d^{\mathrm{sgn}}(\frac{\rho_0-\rho_1}{2})}$. Consequently, the "acceptance probability" of ρ_i is $\Pr[\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\mathrm{sgn}}) \text{ accepts}] = \frac{1}{2}\Big(1+\mathrm{Tr}\Big(P_d^{\mathrm{sgn}}(\frac{\rho_0-\rho_1}{2})\Big)\rho_i\Big)$.

Therefore, for $i \in \{0,1\}$, it suffices to estimate $\mathrm{Tr}\Big(P_d^{\mathrm{sgn}}(\frac{\rho_0-\rho_1}{2})\rho_i\Big) \pm \varepsilon$ with high probability using $O(1/\varepsilon^2)$ sequential repetitions. □

Proof overview: one-sided error scenario

Proof of Theorem 1.2 ①: $\overline{\textsc{CertQSD}_{\mathsf{log}}} \in \mathsf{coRQ_UL}$.

Our construction is mainly based on the previous quantum tester $\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\mathrm{sgn}})$, then achieving perfect completeness by standard techniques.

🔔 We first notice that our space-efficient QSVT in Theorem 4.3 preserves the parity. In particular, the QSVT implementation associated with $\hat{P}_d^{\mathrm{sgn}}$ satisfies $\hat{P}_d^{\mathrm{sgn}}(\mathbf{0}) = \mathbf{0}$.
This enables us to construct the algorithm $\mathcal{A}$ specified below:

- For *yes* instances $(\rho_0 = \rho_1)$, we thus have $\Pr[\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\mathrm{sgn}})$ accepts$] = \frac{1}{2}$. Then we obtain an algorithm $\mathcal{A}$ accept with certainty via *exact amplitude amplification* [Boyer-Brassard-Høyer-Tapp'98, Brassard-Høyer-Mosca-Tapp'02].
- For *no* instances $(\mathrm{td}(\rho_0, \rho_1) \geq \alpha)$, we have

$$\left|\Pr[\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\mathrm{sgn}}) \text{ accepts}] - \tfrac{1}{2}\right| \geq \Omega(\alpha).$$

  By a direct (still, a bit complicated) calculation, we can make sure the algorithm $\mathcal{A}$ accepts w.p. at most $1 - \Omega(\alpha^2)$.

Finally, we conclude a $\mathsf{coRQ_UL}$ containment from $\mathcal{A}$ by applying *error reduction* for $\mathsf{coRQ_UL}$, which can be deduced from our space-efficient QSVT (Theorem 4.3). □

Conclusions and open problems

Take-home messages on our work

1. Space-bounded quantum state certification problems w.r.t. trace distance and Hilbert-Schmidt distance are $\mathsf{coRQ_UL}$-complete (Theorem 1.2).
   This is the *first* family of natural $\mathsf{coRQ_UL}$-complete problem!
2. Space-bounded quantum state testing problems w.r.t. common distance-like measures (i.e., trace distance, squared Hilbert-Schmidt distance, quantum entropy difference, quantum Jensen-Shannon divergence) are BQL-complete (Theorem 1.4).
3. Quantum singular value transformation on bitstring indexed encoding can be done in *quantum logspace*, with a *randomized* classical pre-processing (Theorem 4.3).

Open problems

1. Are there any other applications of space-efficient QSVT?
2. Space-efficient QSVT with $O(d)$ queries instead of $O(d^2\|\mathbf{c}\|_1)$ in Theorem 4.3, as well as make the pre-processing *deterministic* rather than randomized.
   Quantum query complexity lower bound (in *time-efficient scenarios*) $\Omega(d)$ [Montanaro-Shao'24]
3. (Inspired by Tom Gur) Are space-bounded quantum *channel* testings with respect to appropriate distance-like measures also in BQL?

Thanks!

Space-efficient quantum singular value transformation: Proof sketch

Bounded functions. We mainly follow the construction in [MY23]:

Near-minimax approximation by Chebyshev interpolation [Powell'67]

For any continuous function $f\colon [-1,1] \to \mathbb{R}$, if there is a degree-$d$ polynomial P_d satisfying $\max_{x\in[-1,1]} |f(x) - P_d(x)| \leq \epsilon$, then we have a Chebyshev interpolation polynomial $\hat{P}_d := \frac{c_0}{2} + \sum_{k=1}^{d} c_k T_k$, where $c_k := \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\mathrm{d}x$ and T_k is the k-th Chebyshev polynomial (of the first kind), such that $\max_{x\in[-1,1]} |\hat{P}_d(x) - f(x)| \leq O(\epsilon \log d)$.

1. Space-efficient QSVT implementation for $T_k^{(\mathrm{SV})}(\tilde{\Pi}U\Pi)$ [GSLW19]
2. For any bounded functions, any coefficient c_k is *space-efficiently computable* by the standard numerical integral technique. $\longrightarrow$ A careful analysis is required!
3. Implement $\hat{P}_d^{(\mathrm{SV})}(\tilde{\Pi}U\Pi)$ from $T_k^{(\mathrm{SV})}(\tilde{\Pi}U\Pi)$ by LCU [Berry-Childs-Cleve-Kothari-Somma'15]
   $\longrightarrow$ Query complexity $O(d^2)$ and the operator norm of $\hat{P}_d(\tilde{\Pi}U\Pi)$ is at most $\|\mathbf{c}\|_1$
4. Renormalizing the resulting (bitstring indexed) encoding $\hat{P}_d^{(\mathrm{SV})}(\tilde{\Pi}U\Pi)$
   $\longrightarrow$ Query complexity $O(d^2\|\mathbf{c}\|_1)$ where $\|\mathbf{c}\|_1 \leq O(d)$ in general.

Piecewise-smooth functions. We adapt the construction (i.e., a reduction to a linear combination of bounded functions) in [van Apeldoorn-Gilyén-Gribling-de Wolf'20].

🔔 We thus reduce the main challenge to *stochastic matrix powering problem*, essential for the BPL vs. L problem [Saks-Zhou'99, Cohen-Doron-Sberlo-Ta-Shma'23, Putterman-Pyre'23]. ☐