StoqMA meets *distribution testing*

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???

TQC 2021

Why StoqMA is important?

Dichotomy Theorem on

Constraint **S**atisfaction **P**roblem

over boolean domain

[Schaefer'78]

$$(\neg x_1 \vee x_2) \wedge (x_2 \vee \neg x_3 \vee x_4)$$

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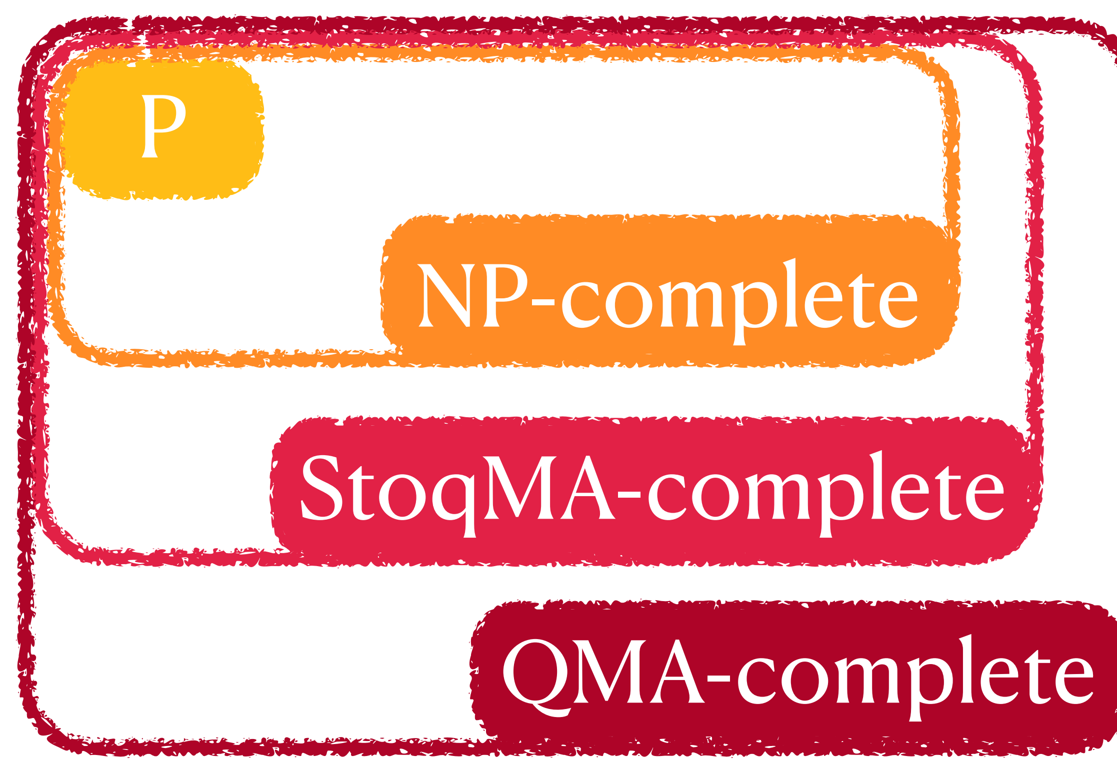
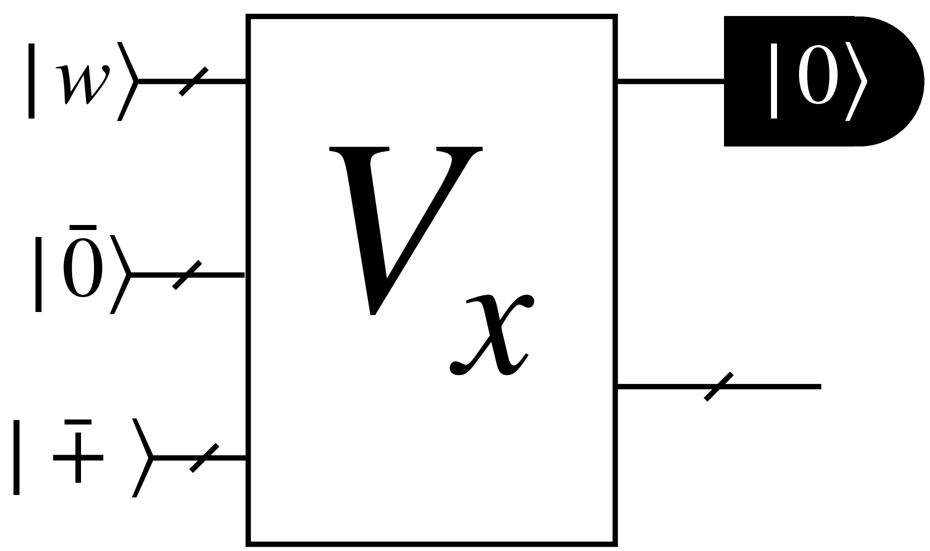


- Assume that $\mathrm{P} \subsetneq \mathrm{NP} \subsetneq \mathrm{StoqMA} \subsetneq \mathrm{QMA}$

What's the complexity class StoqMA?

[Bravyi-Bessen-Terhal'06]

MA



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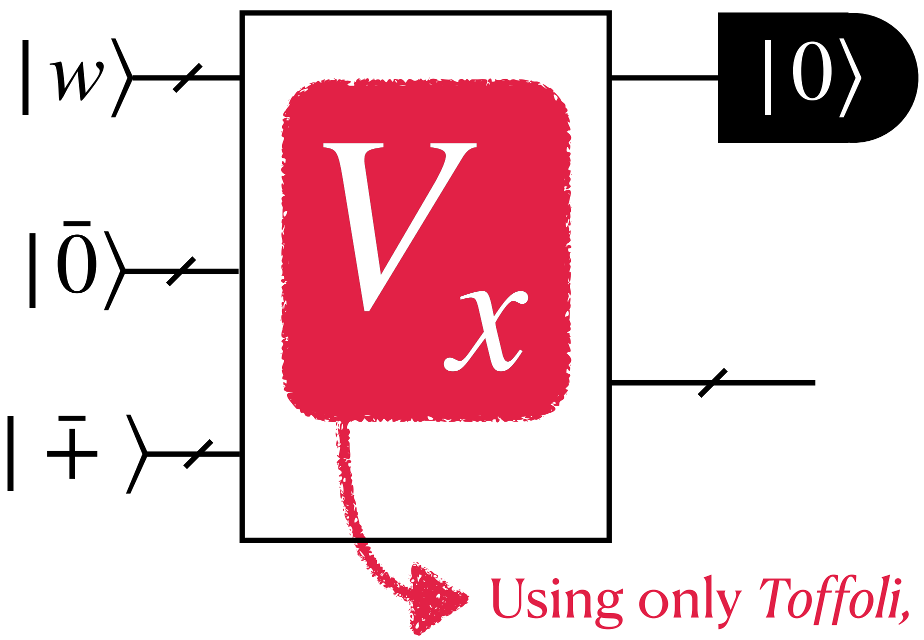

Using only *Toffoli, CNOT, X* gates

- For *yes* instances, $\Pr[V_x \text{ accepts } |w\rangle] \geq a$.
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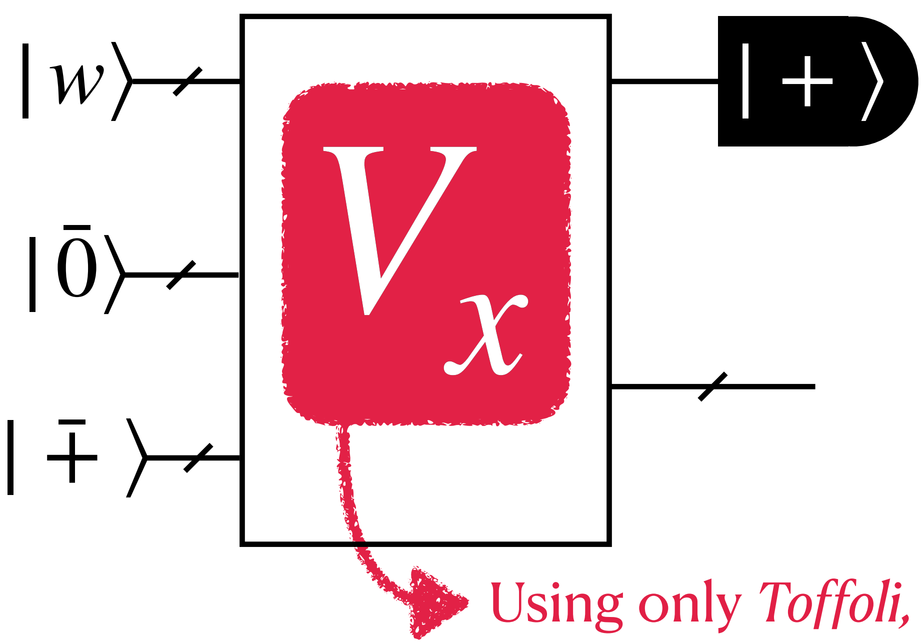

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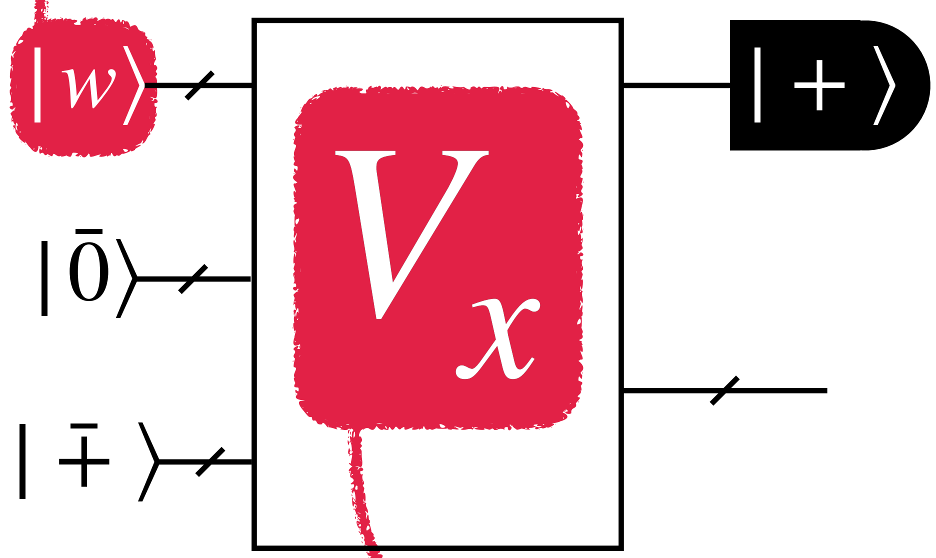

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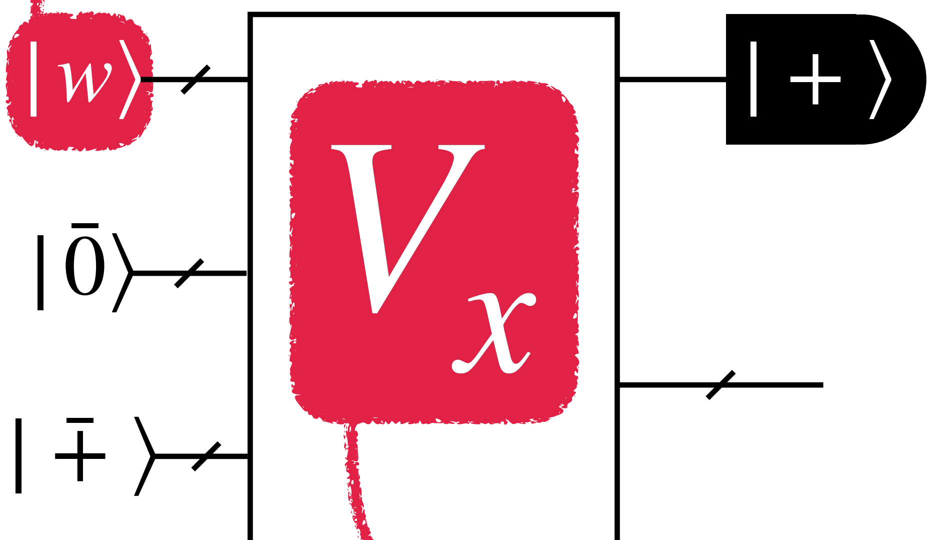

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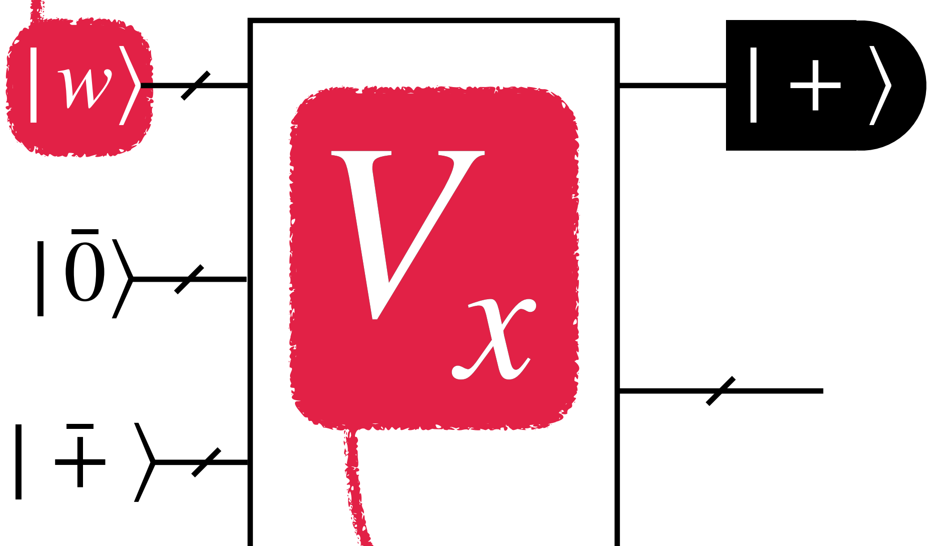

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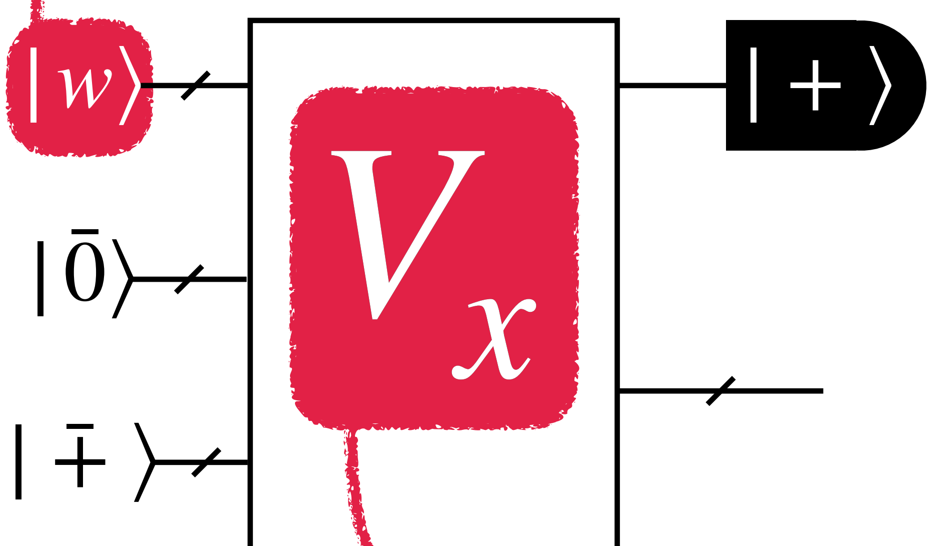

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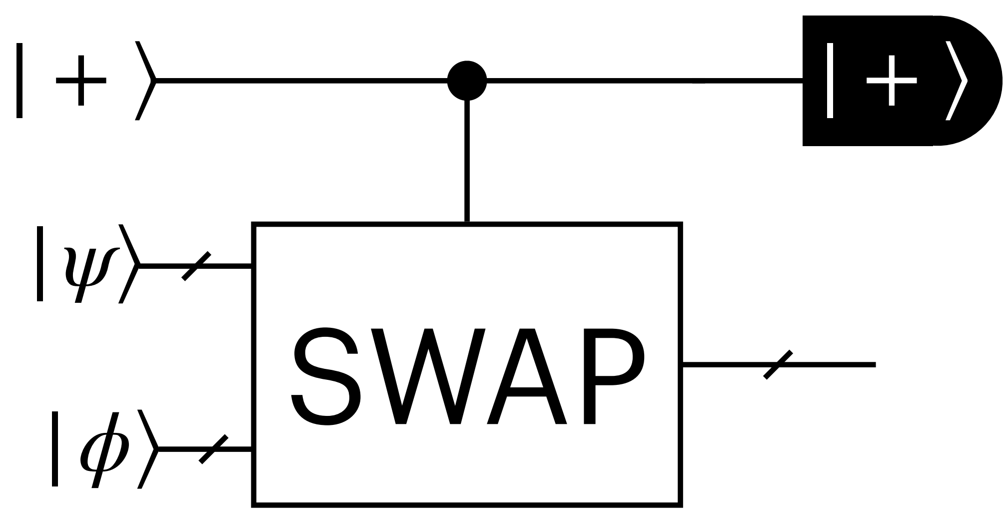
- Definition of StoqMA came from *Stoquastic Local Hamiltonian*.
- MA $\subseteq$ StoqMA $\subseteq$ AM, where AM is two-message randomized generalization of NP.
- Error reduction (i.e., making *a,b* exponentially close to *1* and *1/2*) for StoqMA is *unknown*.

The mystery of the Hadamard-basis measurement

SWAP test



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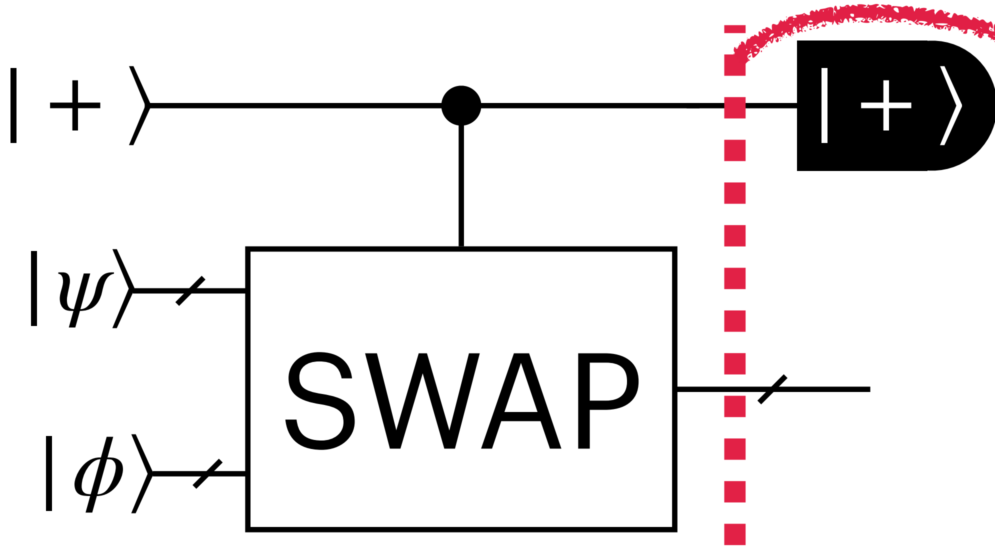

- Dash line: $\sim |0\rangle|\psi\rangle|\phi\rangle + |1\rangle|\phi\rangle|\psi\rangle$
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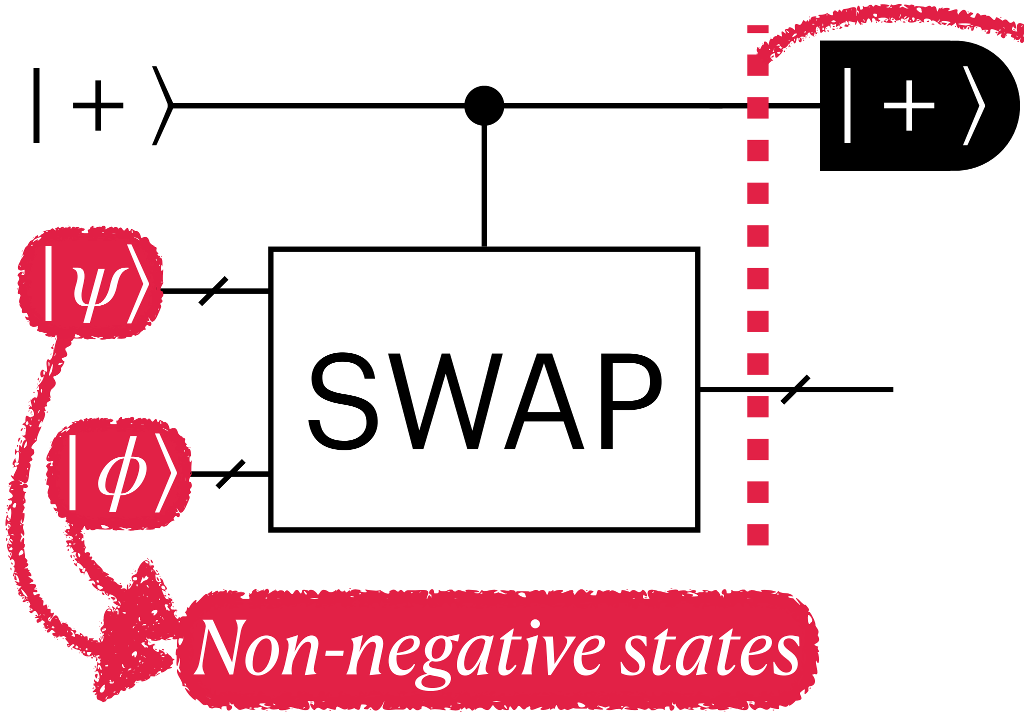

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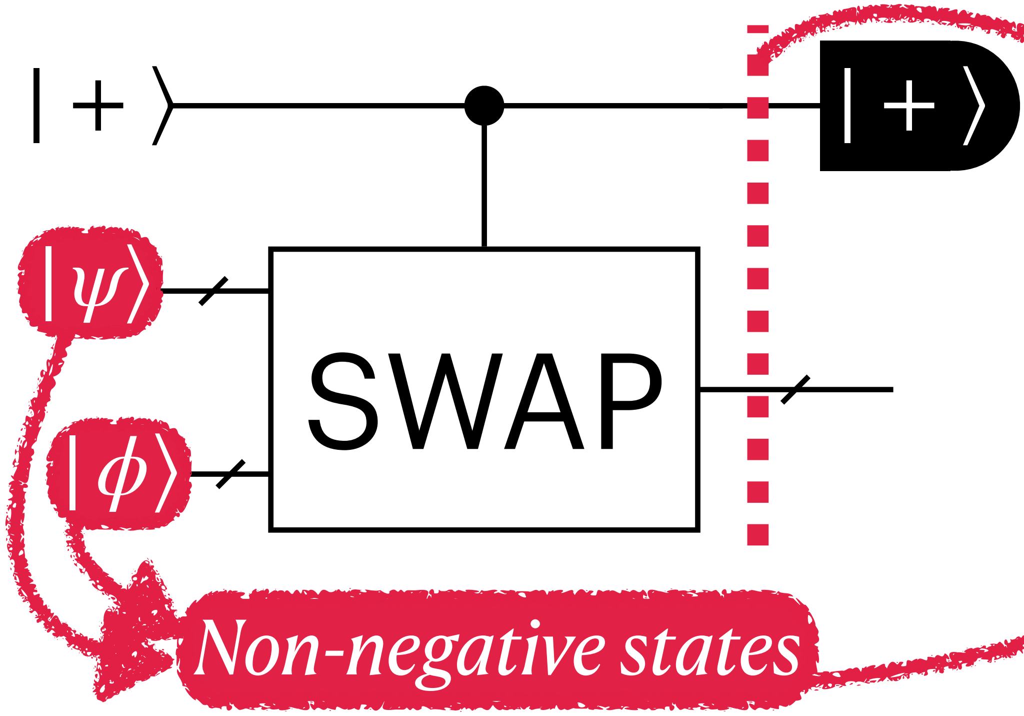
SWAP test as a StoqMA verifier



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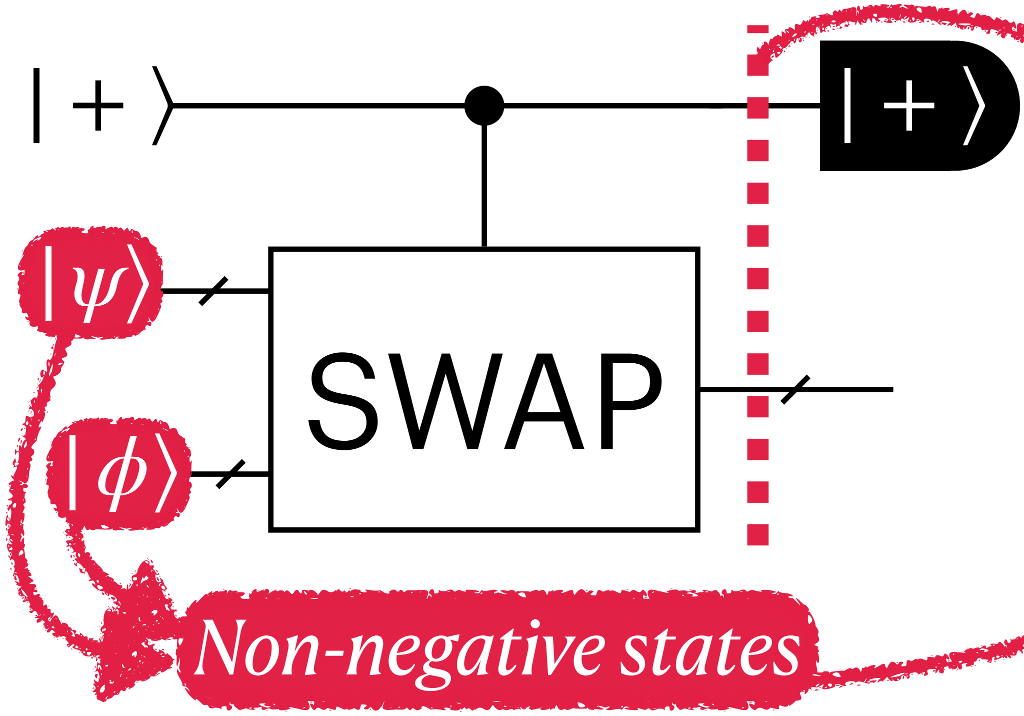

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Take-home message from the SWAP test:

- Single-qubit Hadamard-basis measurement can be thought as a *distribution testing* task!

1st result: eStoqMA $\subseteq$ MA

Easy witness

Consider $|w\rangle = \sum_i \sqrt{p(i)}\,|i\rangle$ such that $\forall x, y \in \{0,1\}^n$, $\frac{p(x)}{p(y)}$ is *efficiently* computable.

- An analogous condition appears in *Guided Stoquastic Local Hamiltonian Problem* [Bravyi'15].

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Proof Sketch. Using the dual access model [Canonne-Rubinfeld'14]:

- *Sample access*: running a copy of V_x and measuring all qubits in computational basis;
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One can approximate max. acc. prob. with *polynomially many of copies* of the witness $|w\rangle$. $\square$

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- **Prop** ([Grilo20]). For any *a,b,* classical-witness-StoqMA$(a, b) \subseteq$ MA$(2a - 1, 2b - 1)$.
- The difficulty of StoqMA roots in *different kinds of optimal witness*!

2^{nd} result: *Distinguishing reversible circuits* with non-negative states

Reversible Circuit Distinguishability is StoqMA-complete

Given *reversible circuits* C_0, C_1, define $|D_k\rangle := C_i |w\rangle |\bar{0}\rangle | \bar{+} \rangle$ for $k = 0,1$ and a *non-negative* witness $|w\rangle$:

- *Yes*: $\exists |w\rangle$ such that $\langle D_0 | D_1 \rangle \geq \alpha$;
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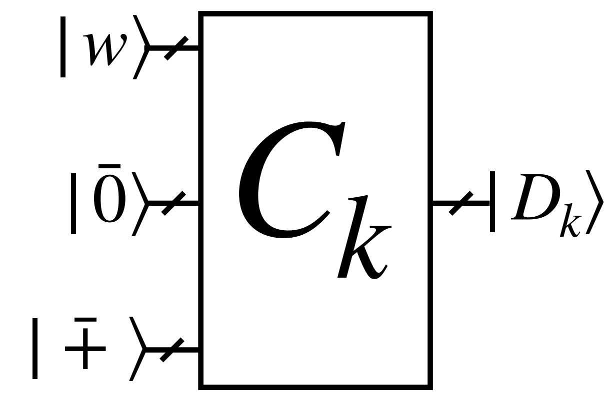

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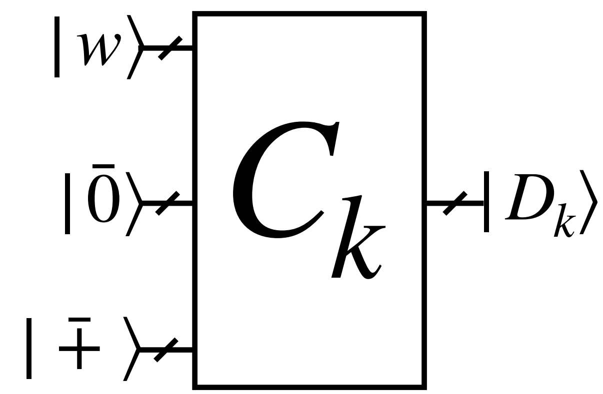

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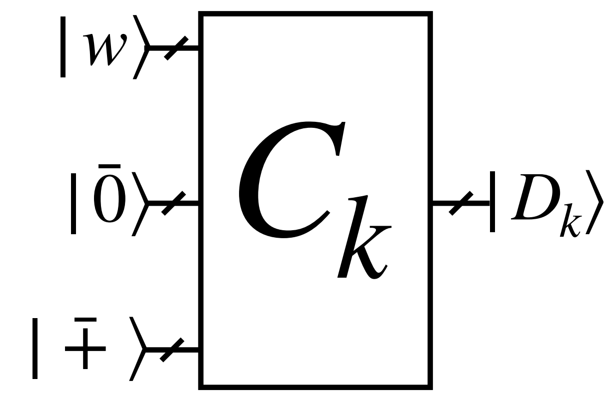

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Soundness error reduction

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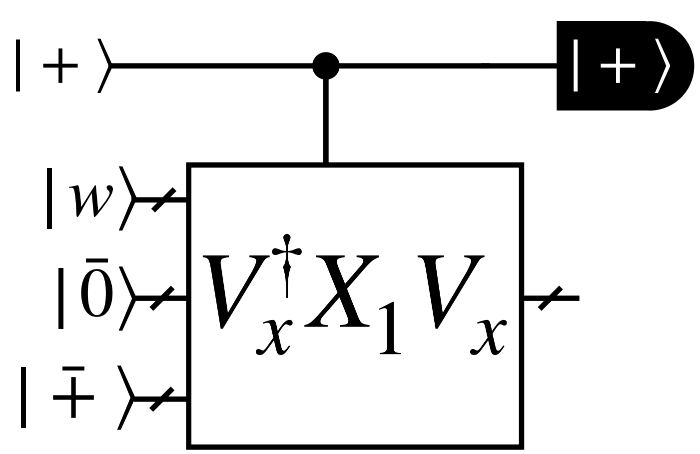
A message from StoqMA-hardness proof



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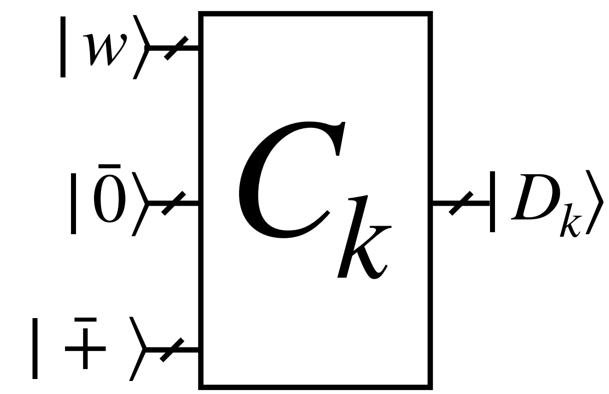

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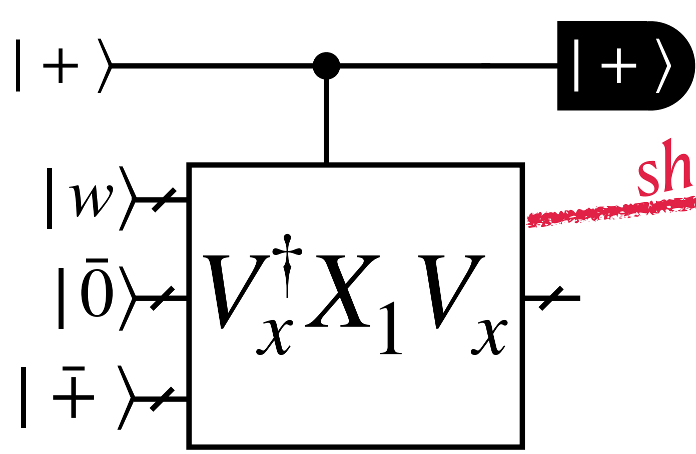
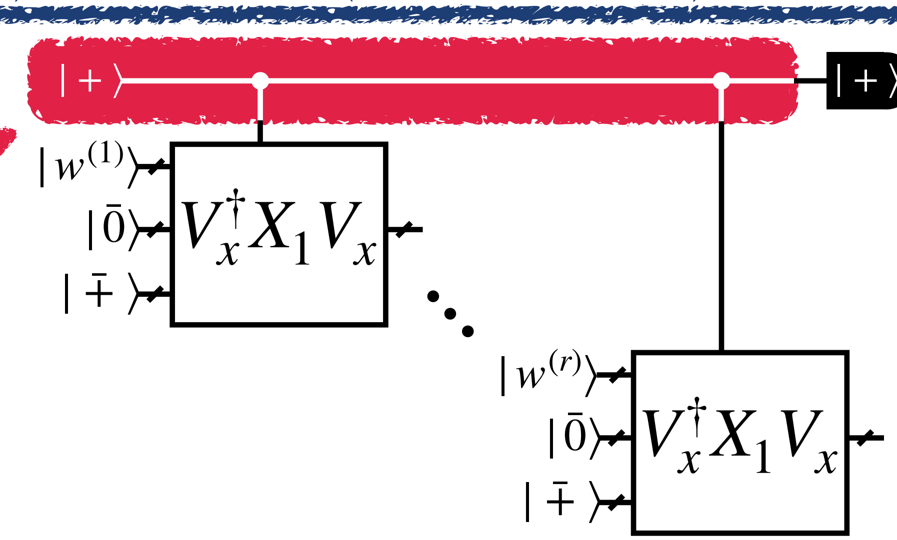


shared control qubit

a special type of parallel repetation



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